Statistics

R.J. Marks II Class Notes

Hogg & Craig

Statistical: Texas Tech University (1977)

I. DISTRIBUTIONS OF RANDOM VARIABLES

A. THE ALGEBRA OF SETS

SUBSET - IF ALL ELEMENTS OF A_1 ARE IN A THEN $A_1 \subseteq A$

NULL SET - THE SET OF NO ELEMENTS

UNION - THE SET OF ALL ELEMENTS BELONGING TO A_1 AND/OR $A_2 \Rightarrow A_1 \cup A_2$

INTERSECTION - THE SET OF ALL ELEMENTS BELONGING TO BOTH A_1 AND $A_2 \Rightarrow A_1 \cap A_2$

SPACE - THE TOTALITY OF ALL ELEMENTS

COMPLEMENT - A^* IS THE SET OF ALL ELEMENTS IN THE SPACE NOT IN A.

B. SET FUNCTIONS

EX: $Q(A) = \int_A f(x)\,dx$

$Q(A) = \Sigma \Sigma_A g(x,y)$

C. THE PROBABILITY SET FUNCTION

$\mathcal{C}$ = SET OF EVERY POSSIBLE OUTCOME OF A RANDOM EXPERIMENT

$C \subseteq \mathcal{C}$

$P(C)$ = PROB. THE OUTCOME IS AN ELEMENT OF C

FUNDEMENTAL AXIOMS

1. $P(C) \geq 0$
2. $P(C_1 \cup C_2 \ldots) = P(C_1) + P(C_2) + \ldots \ni C_i \cap C_j = \phi \ \forall i \neq j$
3. $P(\mathcal{C}) = 1$

$P(C)$ = PROBABILITY SET FUNCTION

THEOREM: $\forall C \subset \mathscr{C}$, $P[C] = 1 - P[C^*]$

PROOF: $C \cup C^* = \mathscr{C}$; $C \cap C^* = \phi$

$\Rightarrow P(C) + P(C^*) = P(\mathscr{C}) = 1$

$\Rightarrow P(C) = 1 - P(C^*)$

THEOREM: $P(\phi) = 0$

THEOREM: IF $C_1 \subset C_2 \subset \mathscr{C}$, THEN

$P(C_1) \leq P(C_2)$

PROOF: $C_2 = C_1 \cup (C_1^* \cap C_2)$

$C_1 \cap (C_1^* \cap C_2) = 0$

$\Rightarrow P(C_2) = P(C_1) + P(C_1^* \cap C_2) \geq P(C_1)$

THEOREM: $\forall C \subset \mathscr{C}$, $0 \leq P(C) \leq 1$

SINCE $\phi \subset C \subset \mathscr{C}$, $0 \leq P(C) \leq 1$

THEOREM: $C_1 \subset \mathscr{C}$, $C_2 \subset \mathscr{C}$

THEN $P[C_1 \cup C_2] = P[C_1] + P(C_2) - P(C_1 \cap C_2)$

D. RANDOM VARIABLES

GIVEN AN EXPERIMENT WITH SAMPLE SPACE $\mathscr{C}$ LET $C \in \mathscr{C}$. A FUNCTION X ASSIGNING ONE AND ONLY ONE REAL NUMBER x TO EACH c (ie $x = X(c)$) IS A RANDOM VARIABLE. THE SPACE OF X IS THE SET OF REAL NUMBERS

$$\mathscr{A} = \{x;\ x = X(c),\ c \in \mathscr{C}\}$$

LET $A \subset \mathscr{A}$

THEN $Pr[X \in A] = P_X(A) = P(C)$

$P_X(A)$ IS AN "INDUCED" PROBABILITY.

GIVEN A RANDOM EXPERIMENT WITH A SAMPLE SPACE $\mathcal{C}$. CONSIDER TWO RANDOM VARIABLES X_1 AND X_2 WHICH ASSIGNS EACH $c \in \mathcal{C}$ A UNIQUE ORDERED PAIR $X_1(c) = x_1$ $X_2(c) = x_2$. THE SPACE OF X_1 AND X_2 IS THE SET OF ORDERED PAIRS

$$\mathcal{A} = [(x_1, x_2);\ x_1 = X_1(c)\ \ x_2 = X_2(c),\ c \in \mathcal{C}]$$

A GENERALIZATION: LET X_i, $i = 1, \ldots n$, BE DEFINED SUCH THAT $X_i(c) = x_i$ IS THE ONE AND ONLY REAL "n-TUPLE" ASSIGNED TO c. THEN THE SPACE OF THESE R.V.'s IS

$$\mathcal{A} = [(x_1 x_2 \ldots x_n):\ x_i = X_i(c),\ i = 1, \ldots, n,\ c \in \mathcal{C}]$$

AND $P_r[(X_1 X_2 \ldots X_n) \in A \subset \mathcal{A}] = P[C]$

$\ni C = [c;\ c \in \mathcal{C}$ AND $X_i(c) \in A]$

E. THE PROBABILITY DENSITY FUNCTION

- DISCRETE

$$f(x) \geq 0 \quad \text{AND} \quad \Sigma_{\mathcal{A}} f(x) = 1$$

THEN $P[A] = P[X \in A] = \Sigma_A f(x)$

- CONTINUOUS

$$f(x) \geq 0 \quad \text{AND} \quad \int_{\mathcal{A}} f(x)\,dx = 1$$

THEN $P[A] = P[X \in A] = \int_A f(x)\,dx$

- MULTIDIMENSIONAL

F. THE DISTRIBUTION FUNCTION

- DEFINITION

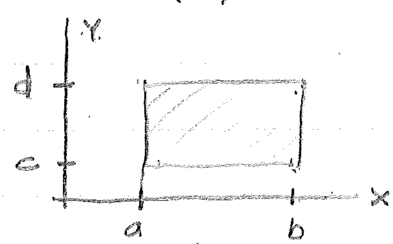
$$F(x) = P[X \le x]$$
$$= \sum_{w \le x} f(w) \quad \Leftarrow \text{DISCRETE}$$
$$= \int_{-\infty}^{x^+} f(\omega)\, d\omega \quad \Leftarrow \text{CONTINUOUS}$$

- PROPERTIES

$$\lim_{x \to \infty} F(x) = 1 \qquad \lim_{x \to -\infty} F(x) = 0$$
$$P_r[a < X \le b] = F(b) - F(a)$$

- TWO DIMENSIONAL

$$\frac{\delta^2}{\delta x\, \delta y} F(x, y) = f(x, y)$$
$$P_r[a < X \le b,\ c < Y \le d]$$
$$= F(b,d) + F(a,c) - F(a,d) - F(b,c)$$



G. CERTAIN PROBABILITY MODELS

- UNIFORM DISTRIBUTION

$$Z = X + Y$$

MISC.

MODE = MAX $f(x)$

"MODE OF DISTRIBUTION" $\Rightarrow$ UNIQUE MODE

MEDIAN x_m: $P_r[X < x_m] \le \frac{1}{2}$

$P_r[X \le x_m] \ge \frac{1}{2}$

"MEDIAN OF DISTRIBUTION" $\Rightarrow$ UNIQUE MEDIAN

H. MATHEMATICAL EXPECTATION

$$E[u(x)] = \int_{-\infty}^{\infty} u(x)\, f(x)\, dx$$
$$\text{OR} \quad \sum_x u(x)\, f(x)$$

MULTI-DIMENSIONAL:

$$E[u(x_1, x_2, \ldots, x_n)] = \int\int\int\cdots\int u(x_1, \ldots x_n)\, f(x_1, \ldots x_n)\, dx_1 \ldots d$$

I. SOME SPECIAL MATHEMATICAL EXPECTATIONS

$\mu = E[X] \Leftarrow$ MEAN

$\sigma^2 = E[(x-\mu)^2] \Leftarrow$ VARIANCE

$\sigma = \sqrt{\sigma^2} \Leftarrow$ STANDARD DEVIATION

$M(t) = E[e^{tx}] \Leftarrow$ MOMENT GEN. FUNCTION

$$E[X^n] = \frac{d^n}{dt^n} M(t)\Big|_{t=0}$$

J. CHEBYCHEV'S INEQUALITY

- GENERAL THEOREM

IF $u(X) \geq 0$, THEN, $\forall\, c > 0$

$\Rightarrow$ $$Pr[u(X) \geq c] \leq \frac{1}{c} E[u(X)]$$

PROOF: LET $A = \{x ; u(x) \geq c\}$

$$0 < E[u(x)] = \int_{-\infty}^{\infty} u(x) f(x) dx = \int_A u(x) f(x) dx + \int_{A^*} u(x) f(x) dx$$

$$\Rightarrow E[u(x)] \geq \int_A u(x)\, f(x) dx$$

BUT, $\forall\, x \subset A$, $u(x) \geq c$. THUS

$$\int_A u(x)\, f(x)\, dx \geq c \int_A f(x) dx = c\, P_r[u(x) \geq c]$$

$$\Rightarrow P_r[u(x) \geq c] \leq \frac{1}{c} E[u(x)]$$

- CHEBYCHEV'S INEQUALITY

$\Rightarrow$ $$P_r[|X-\mu| \geq k\sigma] \leq \frac{1}{k^2} \quad ; k > 0$$

PROOF: IN ABOVE THEOREM, LET

$$u(X) = (X-\mu)^2, \quad c = \sigma^2 k^2$$

$$\Rightarrow P_r[(x-\mu)^2 \geq k^2 \sigma^2] \leq \frac{1}{\sigma^2 k^2} E[(x-\mu)^2]$$

OR

$$P_r[|X-\mu| \geq k\sigma] \leq \frac{1}{k^2}$$

II. CONDITIONAL PROBABILITY & STOCHASTIC INDEPENDENCE

A. CONDITIONAL PROBABILITY

$$P(C_2 | C_1) P(C_1) = P(C_1 \cap C_2)$$

BAYE'S FORMULA:

$$\Rightarrow \quad P(C_i | C) = \frac{P(C_i) P(C|C_i)}{\Sigma_m P(C_m) P(C|C_m)}$$

B. MARGINAL & CONDITIONAL DISTRIBUTIONS

$f(x_1 x_2) \Leftarrow$ JOINT DENSITY

$f_1(x_1) = \int_{x_2} f(x_1 x_2) dx_2 \Leftarrow$ MARGINAL DENSITY

$f(x_2 | x_1) = \frac{f(x_1 x_2)}{f(x_1)} \Leftarrow$ CONDITIONAL DENSITY

C. COVARIANCE & CORRELATION COEFFICIENT

$$\text{cov}(X,Y) = E[(X-\mu_x)(Y-\mu_Y)] = E[XY] - \mu_x \mu_Y$$

$$\rho_{12} = \frac{E[(X_1-\mu_x)(Y-\mu_Y)]}{\sigma_x \sigma_Y} = \frac{1}{\sigma_x \sigma_Y} \text{cov}(X,Y)$$

$$|\rho_{12}| \leq 1$$

$$\rho_{12} = 1 \Rightarrow Y = a + bx, \quad b > 0$$

$$\rho_{12} = -1 \Rightarrow Y = a + bx, \quad b < 0$$

RELATION TO MOMENT GENERATING FUNCTION:

$$\frac{\delta^{k+m} M(t_1 t_2)}{\delta t_1^k \, \delta t_2^m} = E[X^k Y^m]$$

D. STOCHASTIC INDEPENDENCE

DEF: S.I. IFF $f(x_1 x_2) = f(x_1) f(x_2)$

THEM 1: S.I. IFF $f(x_1 x_2) = g(x_1) h(x_2)$

$$g(x_1), h(x_2) > 0$$

THEM 2: $P_r[a<x<b, c<Y<d] = P_r[a<x<b] \, P_r[c<x<d]$

THEM 3: $E[u(X_1) V(X_2)] = E[u(X_1)] E[V(X_2)]$

THEM 4: IF S.I, THEN

$$M(t_1, t_2) = M(0, t_2) M(t_1, 0)$$

RANDOM VARIABLES $X_1, X_2, \ldots X_n$ ARE SI IF

$$f(x_1, \ldots x_n) = f(x_1) f(x_2) \ldots f(x_n)$$

III. SOME SPECIAL DISTRIBUTIONS

A. BINOMIAL

n-REPEATED BERNOULLI TRIALS

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$M(t) = [(1-p) + pe^t]^n$$

$$\mu = np \qquad \sigma^2 = np(1-p)$$

B. POISSON

$$f(x) = \frac{\mu^x e^{-\mu}}{x!}$$

$$M(t) = e^{\mu[e^t - 1]}$$

$$\mu = \mu = \sigma^2$$

G. GAMMA AND CHI-SQUARE

$$\Gamma \Rightarrow f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \qquad 0<x<\infty$$

$$M(t) = \frac{1}{(1-\beta t)^\alpha}$$

$$\chi^2_r \Rightarrow f(x) = \frac{1}{\Gamma(\frac{r}{2})2^{r/2}} x^{\frac{r}{2}-1} e^{-x/2}; \quad 0<x<\infty$$

$$M(t) = (1-2t)^{-r/2}$$

$$\mu = r \qquad \sigma^2 = 2r$$

H. NORMAL

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$M(t) = e^{\mu t + b^2 t^2/2}$$

$$(x-\mu)^2/\sigma^2 \sim \chi^2(1)$$

I. BIVARIATE NORMAL

IV. DISTRIBUTIONS OF FUNCTIONS OF R.V.'S

A. DISTRIBUTION FUNCTION TECHNIQUE

$$F(Y) = P_r[U(x_1, \ldots, x_n) \leq Y]$$

WHERE $\quad Y = U(X_1, X_2, \ldots, X_n)$

B. CHANGE OF VARIABLE TECHNIQUE

LET $\quad Y_1 = U_1(X_1, X_2) \qquad Y_2 = U_2(X_1, X_2)$

$X_1 = \omega_1(Y_1, Y_2) \qquad X_2 = \omega_2(Y_1, Y_2)$

$$|J| = \begin{vmatrix} \frac{\delta \omega_1}{\delta Y_1} & \frac{\delta \omega_1}{\delta Y_2} \\ \frac{\delta \omega_2}{\delta Y_1} & \frac{\delta \omega_2}{\delta Y_2} \end{vmatrix}$$

$$\Rightarrow f_Y(Y_1 Y_2) = |J| \, f_X[\omega_1, \omega_2]$$

C. t & F DISTRIBUTIONS

$T = \frac{W}{\sqrt{V/r}} \qquad W \sim n(0,1), \quad V \sim \chi^2(r)$

$F = \frac{U/r_1}{V/r_2} \quad ; U \sim \chi^2(r_1) \qquad V \sim \chi^2(r_2)$

$$f \sim F_{r_1 r_2} \Rightarrow \frac{1}{f} \sim F_{r_2 r_1}$$

D. EXTENSION OF CHANGE OF VARIABLES

$x_1 = \omega_{1i}(Y_1, Y_2 \ldots Y_n)$

$x_2 = \omega_{2i}(Y_1 \ldots Y_n) \qquad i = 1, \ldots, K$

$\vdots$

$X_n = \omega_{ni}(Y_1 \ldots Y_n)$

$$J_i = \begin{vmatrix} \delta\omega_{1i}/\delta Y_1 & \cdots & \delta\omega_{1i}/\delta Y_n \\ & \cdots & \delta\omega_{ni}/\delta Y_n \end{vmatrix}$$

$$\Rightarrow g(Y_1 \ldots Y_n) = \sum_{i=1}^{K} |J_i| \, f[\omega_{1i} \ldots \omega_{ni}]$$

E. MOMENT GENERATING FUNCTION TECHNIQUE

EX. $Y = \sum_1^n k_i X_i \quad , X_i \sim n(\mu, \sigma^2) \Rightarrow Y \sim n(\Sigma k_i \mu, \Sigma k_i^2 \sigma^2)$

EX. $Y = \sum_1^n \left(\frac{X_i - \mu}{\sigma} \right)^2$

$$X_i \sim n(\mu, \sigma^2) \Rightarrow Y \sim \chi^2_n$$

F. DISTRIBUTIONS OF $\bar{X}$ & $\frac{nS^2}{\sigma^2}$

$$x_i \sim n(\mu, \sigma^2) \Rightarrow \bar{X} \sim n(\mu, \frac{\sigma^2}{n})$$

$$nS^2/\sigma^2 \sim \chi^2(n-1)$$

IN GENERAL, FOR INDEPENDENCE

$$\mu_Y = \mu \sum^n k_i \quad \sigma_Y^2 = \sigma^2 \sum^n k_i^2$$

$$\ni Y = \sum^n k_i X_i$$

V. LIMITING DISTRIBUTIONS

A. $\lim_{n \to \infty} F_n$

B. STOCHASTIC CONVERGENCE

Y_n CONVERGES STOCHASTICALLY TO C IFF

$$\lim_{n \to \infty} P_r [|Y_n - C| < \epsilon] = 1$$

USE CHEBYCHEV:

$$P_r [|X - \mu_n| \geq k\sigma_n] \leq \frac{1}{k^2}$$

C. LIMITEING MGF'S

VI. INTERVAL ESTIMATION

A. RANDOM INTERVALS

B. CONFIDENCE INTERVALS FOR MEANS

- NORMAL: $P_r\left[\bar{x} - \frac{2\sigma}{\sqrt{n}} < \mu < \bar{x} + \frac{2\sigma}{\sqrt{n}}\right] = \alpha$

α = CONFIDENCE COEFFICIENT

FOR σ^2 UNKNOWN:

$$T_{r-1} = \frac{n(0,1)}{\sqrt{\chi^2_{r-1}/(r-1)}} = \frac{(\bar{x}-\mu)/\sigma\sqrt{n}}{\sqrt{\frac{ns^2}{\sigma^2(n-1)}}} = \frac{\bar{x}-\mu}{s/\sqrt{n-1}}$$

$$\alpha = P_r\left[-b < T_{r-1} = \frac{\bar{x}-\mu}{s/\sqrt{n-1}} < b\right]$$

$$= P_r\left[\bar{x} - \frac{bs}{\sqrt{n-1}} < \mu < \bar{x} + \frac{bs}{\sqrt{n-1}}\right]$$

- USING CENTRAL LIMIT THEOREM

$$\bar{x} = \Sigma_n \frac{x_i}{n} \qquad s^2 = \frac{1}{n}\sum^n (x_i - \bar{x})^2$$

$$\frac{\bar{x}-\mu}{\sigma} \sim n(0,1) \qquad \frac{ns^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$\frac{\bar{x}-\mu}{s/\sqrt{n-1}} \sim T_{n-1} \sim n(0,1)$$

- BINOMIAL (CONFIDENCE INTERVAL FOR p GIVEN n)

$$X_i \sim b(1,p) \Rightarrow Y = \sum^n x_i \sim b(n,p)$$

$$\frac{Y-np}{\sqrt{npq}} \sim n(0,1) \quad \text{(APPROXIMATELY)}$$

$$p \cong \frac{Y}{n} \Longrightarrow \frac{Y-np}{\sqrt{n(\frac{Y}{n})(1-\frac{Y}{n})}} = \frac{Y/n - p}{\sqrt{\frac{Y}{n}(1-\frac{Y}{n})/n}} \sim n(0,1)$$

$$\Rightarrow \alpha = P_r\left[\frac{Y}{n} - z\sqrt{\frac{Y/n(1-Y/n)}{n}} < p < \frac{Y}{n} + z\sqrt{\frac{Y/n(1-Y/n)}{n}}\right]$$

- ANOTHER WAY (NOT IN BOOK)

$$\frac{Y-np}{\sqrt{np(1-p)}} \sim z$$

$$\alpha = P_r\left[-b < \frac{Y-np}{\sqrt{np(1-p)}} < b\right]$$

$$= P_r\left[0 < \frac{(Y-np)^2}{np(1-p)} < b^2\right]$$

MAY SOLVE FOR p WITH QUADRATIC FORMULA TO ESTABLISH CONFIDENCE INTERVAL BOUNDS.

- FOR EXACT CONFIDENCE INTERVALS, WE MUST USE BINOMIAL TABLES.

C. CONFIDENCE INTERVALS FOR DIFFERENCES OF MEAN:

- NORMAL: $X_i \sim n(\mu_1, \sigma^2)$ $\quad Y_i \sim n(\mu_2, \sigma^2)$

$$\bar{X} - \bar{Y} \sim n\left(\mu_1 - \mu_2, \frac{\sigma^2}{n} + \frac{\sigma^2}{m}\right)$$

$$\frac{n s_1^2}{\sigma^2} \sim \chi^2_{n-1} \qquad \frac{m s_2^2}{\sigma^2} \sim \chi^2_{m-1}$$

$$\Rightarrow \frac{n s_1^2 + m s_2^2}{\sigma^2} \sim \chi^2_{n+m-2}$$

THUS $$\frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\left(\frac{n s_1^2 + m s_2^2}{n - m - 2}\right)\left(\frac{1}{n} + \frac{1}{m}\right)}} \sim T_{n+m-2}$$

- POISSON (USING CENTRAL LIMIT THEOREM)

$$\frac{\left(\frac{Y_1}{n_1} - \frac{Y_2}{n_2}\right) - (\mu_1 - \mu_2)}{[p_1(1-p_1)/n_1 + p_2(1-p_2)/n_2]^{\frac{1}{2}}}$$

NOW $$W = \frac{\left(\frac{Y_1}{n_1} - \frac{Y_2}{n_2}\right) - (\mu_1 - \mu_2)}{U} \sim n(0,1)$$

$$\ni U = \left[\left(\frac{Y_1}{n_1}\right)\left(1 - \frac{Y_1}{n_1}\right)/n_1 + \left(\frac{Y_2}{n_2}\right)\left(1 - \frac{Y_2}{n_2}\right)/n_2\right]^{\frac{1}{2}}$$

$$\Rightarrow \alpha = P_r\left[\frac{Y_1}{n_1} - \frac{Y_2}{n_2} - bU < p_1 - p_2 < \frac{Y_1}{n_1} - \frac{Y_2}{n_2} + bU\right]$$

D. CONFIDENCE INTERVALS FOR VARIANCES

$x \sim n(\mu, \sigma^2)$

- μ KNOWN $\Rightarrow Y = \frac{1}{\sigma^2}\sum^n (x_i - \mu)^2 \sim \chi^2(n)$

$$P_r[a < \chi^2_n < b] = \alpha$$

- μ UNKNOWN $\Rightarrow S^2 = \frac{1}{n}\sum^n (x_i - \bar{x})^2$ ~~$\sim \chi^2_{n-1}$~~

$$\Rightarrow \alpha = P_r\left[\frac{nS^2}{b} < \sigma^2 < \frac{nS^2}{a}\right]$$

- FOR RATIO OF σ_1^2/σ_2^2

$$\frac{n s_1^2}{\sigma_1^2} \sim \chi^2_{n-1} \qquad \frac{m s_2^2}{\sigma_2^2} \sim \chi^2_{m-1}$$

$$F_{n-1,m-1} = \frac{n s_1^2}{\sigma_1^2(n-1)} \Big/ \frac{m s_2^2}{\sigma_2^2(m-1)}$$

$$\alpha = P_r[a < F_{n-1,m-1} < b]$$

$$= P_r\left[a \frac{m s_2^2}{(m-1)} \Big/ \frac{n s_1^2}{(n-1)} < \frac{\sigma_2^2}{\sigma_1^2} < b \frac{m s_2^2}{(m-1)} \Big/ \frac{n s_2^2}{(n-1)}\right]$$

E. BAYESIAN INTERVAL ESTIMATION

$X \sim g(Y/\Theta)$

$\Theta \sim h(\Theta)$ = PRIOR pdf

$k(\Theta, Y) = h(\Theta)\, g(Y/\Theta)$

$k_1(Y) = \int h(\Theta)\, g(Y/\Theta)\, d\Theta$

$k(\Theta/Y) = \frac{1}{k_1(Y)}\, h(\Theta)\, g(Y/\Theta)$

$\Rightarrow P_r\left[u(Y) < \Theta < v(Y) \mid \mathbf{Y} = Y \right]$

$= \int_{u(Y)}^{v(Y)} k(\Theta/Y)\, d\Theta$

$k(\Theta/Y)$ = POSTERIOR pdf

VII. POINT ESTIMATION AND SUFFICIENT STATISTICS

A. THE PROBLEM OF POINT ESTIMATION

- UNBIASED STATISTIC $\hat{\theta}$

  $E[\hat{\theta}] = \theta$

- BEST STATISTIC IS UNBIASED AND

  $\mathrm{Var}\,\hat{\theta} \le \mathrm{Var}$[ANY OTHER UNBIASED STATISTIC]

B. A SUFFICIENT STATISTIC FOR A PARAMETER

- DEFN: $Y_1 = u(x_1 x_2 \ldots x_n)$ IS SUFFICIENT STATISTIC FOR θ IFF $\forall$ $Y_2 = u_2(x_1 \ldots x_n) \ldots Y_n = u_n(x_1 \ldots x_n)$

  $h(Y_2 \ldots Y_n / Y_1)$ DOES NOT DEPEND ON θ.

- FISHER-NEYMAN CRITERION (FACTORIZATION THEM)

  $Y_1 = u_1(X_1, X_2 \ldots X_n) \sim g_1(Y_1, \theta)$

  Y_1 IS SUFF. IFF: $f(x_1;\theta) f(x_2;\theta) \ldots f(x_n;\theta)$

  $= L(x_1 \ldots x_n;\theta) = g_1(u_1(x_1 \ldots x_n);\theta)\, H(x_1 \ldots x_n)$

  (USE WHEN DOMAIN DEPENDS ON θ)

- Y_1 IS SUFF IFF

  $L(x_1 \ldots x_n;\theta) = k_1[u_1(x_1 \ldots x_n);\theta]\, k_2(x_1 \ldots x_n)$

  $\ni$ k_1 & k_2 ARE NON-NEGATIVE

  (USE WHEN DOMAIN IS PARAMETER INDEPENDENT)

C. THE RAO-BLACKWELL THEOREM

- LET X & Y DENOTE RANDOM VARIABLES $\ni$ Y HAS MEAN μ AND VARIANCE σ_Y^2. LET $E[Y/X] = \phi(x)$. THEN $E[\phi(x)] = \mu$

  AND $\mathrm{Var}[\phi(x)] \le \mathrm{Var}[Y]$

- THEOREM: LET $Y_1 = u_1(x_1 \ldots x_n)$ BE SUFFICIENT FOR θ AND $Y_2 = u_2(x_1 \ldots x_n)$ BE UNBIASED & NOT A FUNCTION OF Y_1 ALONE. THEN $\phi(Y_1) = E[Y_2/Y_1]$ IS A STATISTIC $\ni$ $\phi(Y_1)$ IS UNBIASED & SUFF & $\mathrm{Var}[\phi(Y_1)] < \mathrm{Var}\, Y_2$

D. COMPLETENESS: $(f(x;\theta);\theta\in\Omega)$. LET $u(x)$ BE A CONTINUOUS FUNCTION OF X (AND NOT θ). IF THE CONDITION $E[u(x)]=0$ REQUIRES $u(x)=0$, THE FAMILY OF DENSITY FUNCTIONS IS COMPLETE.

E. UNIQUENESS (LEHMANN-SCHEFFÉ THEOREM)

LET $Y_1=u_1(X_1\ldots X_n)$ BE SUFFICIENT FOR θ. IF $\exists$ A CONTINUOUS FUNCTION OF $Y_1 \ni$ $E[\phi(Y_1)]=\theta$, THEN $\phi(Y_1)$ IS THE BEST STATISTIC FOR θ.

F. EXPONENTIAL CLASS OF pdf's

- $f(x;\theta)=\exp[p(\theta)K(x)+S(x)+q(\theta)]$; $a<x<b$; $\gamma<\theta<\delta$

  $Y_1=\sum^n K(x_i)$ IS SUFF. FOR θ

  $g_1(Y_1;\theta)=R(Y_1)e^{p(\theta)Y_1+nq(\theta)}$ IS COMPLETE

G. FUNCTIONS OF A PARAMETER

MAY RESTRICT ATTENTION TO FUNCTIONS OF THE SUFFICIENT STATISTIC

H. SEVERAL PARAMETERS

- DEF: $f(x;\theta_1,\theta_2)$,

  $Y_i=u_i(x_1\ldots x_n)$

  Y_1 AND Y_2 ARE JOINTLY SUFFICIENT IF

  $$h(Y_3,\ldots Y_n/Y_1,Y_2)=\frac{g(Y_1\ldots Y_n;\theta_1\theta_2)}{g_{12}(Y_1,Y_2;\theta_1,\theta_2)}$$

  ie, h IS NOT DEPEND ON θ.

THEM: $L(x_1\ldots x_n;\theta_1\theta_2)=g_{12}[u_1(x_1\ldots x_n),u_2(x_1\ldots x_n);\theta_1\theta_2]$
$\times H(x_1\ldots x_n)$

OR $L(x_1\ldots x_n;\theta_1,\theta_2]$
$=k_1[u_1(x_1\ldots x_n),u_2(x_1\ldots x_n);\theta_1\theta_2]k_2(x_1\ldots x_n)$

MULTIPARAMETER EXPONENTIAL CLASS

$f(x;\theta_1 \ldots \theta_m) = \exp\left[\sum_{j=1}^{m} p_j(\theta_1 \ldots \theta_m) K_j(x) + S(x) + q(\theta_1 \ldots \theta_m)\right]$

$Y_j = \sum_{i=1}^{n} K_j(x_i)$; $j = 1, \ldots m$ ARE SUFF.

I. SUFFICIENCY, COMPLETENESS, & STOCHASTIC INDEPENDENCE

- THEOREM: LET $X_i \sim f(x_i;\theta)$ & $Y_1 = U_1(X_1 \ldots X_n)$ BE SUFFICIENT FOR θ. & $g_1(Y_1;\theta)$ BE COMPLETE. LET $Z = U(X_1 \ldots X_n) \neq V(Y_1)$. IF DISTRIBUTION OF Z DOES NOT DEPEND ON θ, Z & Y_1 ARE INDEP. (ALSO, IF $M_Z(t)$ DOES NOT DEPEND ON θ)

VIII. FURTHER TOPICS IN POINT ESTIMATION

A. THE CRAMÉR-RAO INEQUALITY

- IF Y IS AN UNBIASED STATISTIC FOR θ:

  $$\sigma_Y^2 \geq \frac{1}{n E[\{\frac{\partial}{\partial \theta} \ln f(x;\theta)\}^2]} = \frac{1}{-n E[\frac{d^2}{d\theta^2} \ln f(x;\theta)]}$$

- Y IS EFFICIENT IF σ_Y^2 MEETS CRAMER-RAO BOUND
- EFFECIENCY $= \frac{\text{CRAMER-RAO BOUND}}{\sigma_Y^2}$
- A STATISTIC THAT CONVERGES STOCHASTICALLY TO A PARAMETER θ IS CONSISTANT

B. MAXIMUM LIKELIHOOD ESTIMATION OF PARAMETERS

- LIKELIHOOD FUNCTION:

  $$L(\theta; x_1, \ldots x_n) \triangleq f(x_1;\theta) \cdots f(x_n;\theta) \quad ; \theta \in \Omega$$

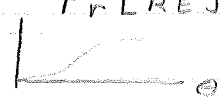
  MAXIMIZE L (OR $\ln L$) WITH RESPECT TO θ & CALL IT $\hat{\theta}$

- THEOREM: IF Y IS SUFF. FOR θ AND $\hat{\theta}$ IS M.L.E. OF θ, IS UNIQUE, THEN $\hat{\theta} = \hat{\theta}(Y)$

IX. STATISTICAL HYPOTHESIS

A. SOME DEFINITIONS

- STATISTICAL HYPOTHESIS: ASSERTATION ABOUT DISTRIBUTION OF ONE OR MORE R.V.'S
  - SIMPLE: HYPOTHESIS SPECIFIES DISTRIBUTION
  - COMPOUND: DOES NOT
- CRITICAL REGION: PORTION OF SAMPLE SPACE IN WHICH HYPOTHESIS IS REJECTED
- POWER FUNCTION: $P_r[\text{REJECTING } \overset{(H_1)}{H_0} / \theta]$



- SIGNIFICANCE LEVEL: $\alpha = P_r[H_1 / H_0]$
- POWER: $\beta = P_r[H_1 / H_1]$

B. CERTAIN BEST TESTS

- DEF: C IS BEST CRITICAL REGION OF SIZE α FOR $H_0: \theta = \theta'$ VS $H_1: \theta = \theta''$ IF β IS MAX
- NEYMAN-PEARSON THEOREM

$$L(\theta; x_1, \ldots x_n) = f(x_1; \theta) \ldots f(x_n; \theta)$$

THEN, BEST TEST IS

$$\frac{L(\theta'; x_1 \ldots x_n)}{L(\theta''; x_1 \ldots x_n)} \underset{H_0}{\overset{H_1}{\lesseqgtr}} K \qquad H_0: \theta = \theta'; \; H_1: \theta = \theta''$$

C. UMP TESTS (UNIFORMLY MOST POWERFUL)

- A CRITICAL REGION C IS UMP AT SIZE α IF C IS BEST CRITICAL REGION FOR TESTING H_0 AGAINST EACH SIMPLE HYPOTHESIS IN H_1.

X. OTHER STATISTICAL TESTS

A. LIKELIHOOD RATIO TESTS

$H_0: \theta_1 \in \omega$

$H_1: \theta \in \Omega \quad ; \omega \in \Omega$

$$\lambda = L(\hat{\omega}) / L(\hat{\Omega}) \geq 0$$

$$\lambda \underset{H_0}{\overset{H_1}{\lessgtr}} \lambda_0$$

B. CHI-SQUARE TESTS

$X_i \sim n(\mu_i, \sigma_i^2)$

$$\frac{1}{\sigma_1 \sigma_2 \cdots \sigma_n (2\pi)^{n/2}} e^{-\frac{1}{2} \sum^n \left(\frac{x_i - \mu_i}{\sigma_i} \right)^2}$$

$$\sum^n (x_i - \mu_i)^2 / \sigma_i^2 \sim \chi^2_n$$

G. WATKINS AND SHUCANY, COMMUNICATIONS IN STATISTICS $\underline{2}$ #4, 285 (1973)

LET $X \sim N(\mu, \sigma^2)$
WE WISH TO FIND THE "BEST" STATISTIC FOR $f(\mu)$. CALL IT $\hat{f}(\mu)$

CASE 1: σ^2 KNOWN

$$\hat{f}(\mu) = f(\bar{x}) + \sum_{k=1}^{\infty} \frac{(-1)^k f^{(2k)}(\bar{x})}{k!} \left(\frac{\sigma^2}{2n}\right)^k$$

CASE 2: σ^2 NOT KNOWN

$$\hat{f}(\mu) = f(\bar{x}) + \sum_{m=1}^{\infty} \frac{(-1)^m f^{(2m)}(\bar{x})}{\left(\frac{n-1}{2}\right)_m m!} \left(\frac{S^2}{4}\right)^m$$

$$S^2 = \frac{\sum^n (x-\bar{x})^2}{n}$$

$$(\alpha)_m = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}$$

FINAL PLUG SHEET

MODE = max $f(x)$

MEDIAN x_m: $Pr[X < x_m] \le \frac{1}{2}$ $\quad Pr[X \ge x_m] \ge \frac{1}{2}$

INEQUALITY: $Pr[u(x) \ge c] \le \frac{1}{c} E[u(x)]$; $u(x) \ge 0$

CHEBYCHEV'S INEQUALITY: $Pr[|x-\mu| \ge k\sigma] \le \frac{1}{k^2}$

BAYE'S FORMULA: $P(C_i|C) = \dfrac{P(C_i)P(C|C_i)}{\sum_m P(C_m)P(C|C_m)}$

COVARIANCE: $cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

CORRELATION COEFFICIENT: $\rho_{12} = \dfrac{cov(X,Y)}{\sigma_X \sigma_Y}$

POISSON: $f(x) = \dfrac{\mu^x e^{-\mu}}{x!}$; $M(t) = e^{\mu[e^t - 1]}$; $\mu = \sigma^2$

χ^2: $f(x) = \dfrac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$; $M(t) = \dfrac{1}{(1-\beta t)^\alpha}$, $\mu = r, \sigma^2 = 2r$

T : $T_r = n(0,1) \Big/ \sqrt{\chi^2_r / r}$

F : $F_{r_1 r_2} = \chi^2_{r_1}/r_1 \Big/ \chi^2_{r_2}/r_2 = \dfrac{1}{F_{r_2 r_1}}$

IF $X_i \sim n(\mu, \sigma) \Rightarrow \bar{X} \sim n(\mu, \frac{\sigma^2}{n})$ $\quad \frac{nS^2}{\sigma^2} \sim \chi^2_{n-1}$

STOCHASTIC CONVERGENCE: $\lim_{n\to\infty} Pr[|Y_n - c| < \epsilon] = 1$

IF $X_i \sim n(\mu,\sigma)$, $\dfrac{\bar{X}-\mu}{S/\sqrt{n-1}} \sim T_{n-1}$

SUFFICIENT STAT: $L(x_1 \ldots x_n; \theta) = g_1[u_1(x_1 \ldots x_n); \theta] H(x_1 \ldots x_n)$

OR $L(x_1 \ldots x_n; \theta) = k_1[u_1(x_1 \ldots x_n); \theta] k_2(x_1 \ldots x_n)$

RAO-BLACKWELL THEM: Y_1 SUFF & Y_2 UNBIASED & $\phi(Y_1) = E(Y_2/Y_1)$

THEN $E[\phi(Y_1)] = \theta$ AND $Var[\phi(Y_1)] < Var\, Y_2$

COMPLETENESS: $E[u(x)] = 0 \Rightarrow u(x) = 0$

UNIQUENESS: IF Y_1 IS SUF & $\phi(Y_1)$ UNBIASED $\Rightarrow \phi(Y_1)$ BEST

EXPONENTIAL CLASS: $f(x;\theta) = e^{p(\theta)K(x) + q(\theta) + S(x)}$

$Y_1 = \sum^n K(X_i)$ SUFF & $g(Y_1) = R(Y_1)\, e^{p(\theta)Y_1 + nq(\theta)}$ (COMPLETE)

INDEPENDENT: Y SUFF. IF $g(z)$ IS NOT θ DEP, Z & Y ARE IND

CRAMER RAO INEQUALITY : Y IS UNBIASED

$\sigma_Y^2 \ge \dfrac{1}{nE[\{\frac{d}{d\theta} \ln f(x;\theta)\}^2]} = \dfrac{1}{-nE[\frac{d^2}{d\theta^2} \ln f(x;\theta)]}$

MAXIMUM LIKELIHOOD: $L(x_1 \ldots x_n; \theta) = f(x_1 \theta) \ldots f(x_n; \theta)$

IF Y IS SUFF THEN $\hat{\theta} = \hat{\theta}(Y)$

POWER FUNCTION: $Pr[H_1/\theta]$

SIGNIFICANCE LEVEL: $Pr[H_1/H_0] = \alpha$

POWER: $\beta = Pr[H_1/H_1]$

NEYMAN-PEARSON: $\frac{L(\theta'; x_1 \ldots x_n)}{L(\theta''; x_1 \ldots x_n)} \lessgtr_{H_0}^{H_1} k$

LIKELIHOOD RATIO TEST: $H_0: \theta \in \omega \cdot H_1: \theta \in \Omega$

$\lambda = L(\hat{\omega}) / L(\hat{\Omega}) \lessgtr_{H_0}^{H_1} \lambda_0$

BOB MARKS

(1.1) a) $\mathcal{C} = (H, T)$, $C = T$

$P[T] = 1/2$

b) $\mathcal{C} = (1, 2, 3, 4, 5, 6)$

$C = (5, 6)$

$P[C] = 2/6 = 1/3$

c) $\mathcal{C} = (AH, 2H, \ldots QH, KH,$

$AS, 2S, \ldots, KS,$

$AC, 2C, \ldots, KS,$

$AD, 2D, \ldots, KD)$

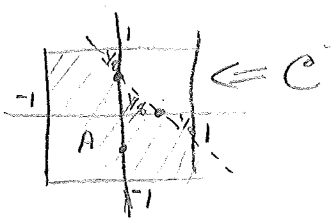
$C = (AS, 2S, \ldots, KS)$

$P(C) = 1/4$

d) $\mathcal{C} = \{X\} \ni 0 \le X \le 1$

$C = \{Y\} \ni 0 \le X \le 1/3$

$P[C] = 1/3$

e) 

$C \Rightarrow X + Y \le \frac{1}{2} \Rightarrow Y \le \frac{1}{2} - X$

$P[C] = \frac{AREA(A)}{AREA(\mathcal{C})} = \frac{7/8 + 7/8 + \frac{1}{8} + 1}{4}$

$= \frac{23}{32}$

(1-2) a. $A_1 = \{x;\ x = 0, 1, 2\}$

$A_2 = \{x;\ x = 2, 3, 4\}$

$A_1 \cup A_2 = \{x;\ x = 0, 1, 2, 3, 4\}$

$A_1 \cap A_2 = \{x;\ x = 2\}$

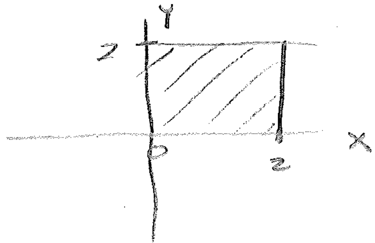
b. $A_1 = \{x;\ 0 < x < 2\}$

$A_2 = \{x;\ 1 \le x \le 3\}$

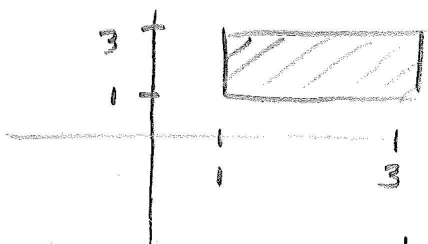
$A_1 \cup A_2 = \{x;\ 0 < x < 3\}$

$A_1 \cap A_2 = \{x;\ 1 \le x < 2\}$

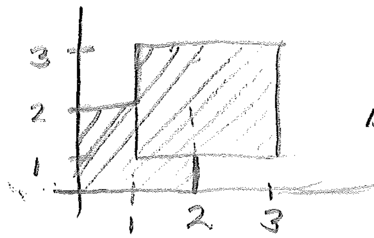
c. $A_1 = \{(x, y);\ 0 < x < 2,\ 0 < y < 2\}$



$A_2 = \{(x, y);\ 1 < x < 3,\ 1 < y < 3\}$

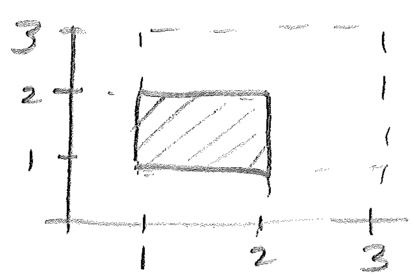


$A_1 \cup A_2 \implies$



THE BOUNDRIES ARE <u>NOT</u> INCLUDED

$A_1 \cap A_2 \implies$

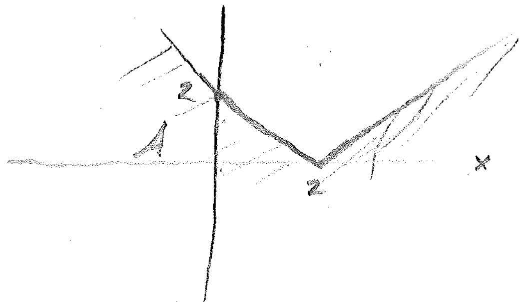


$A_1 \cap A_2 = \{(x, y);\ 1 < x < 2,\ 1 < y < 2\}$

(1.3) (a) $A^* = \{x;\ 0 < x < 5/8\}$

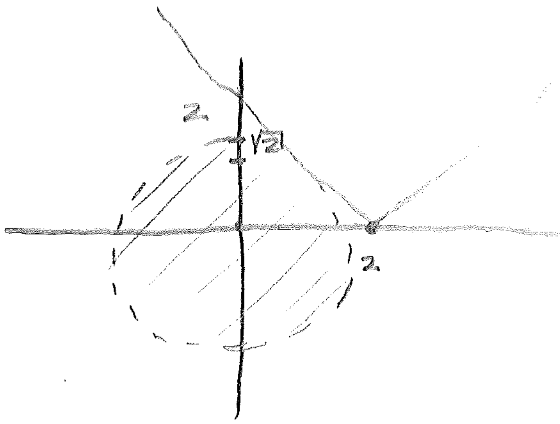
(b) $A^* = \{(x, y, z);\ x^2 + y^2 + z^2 < 1\}$

(c) $A = \{(x, y);\ |x| + |y| \le 2\}$

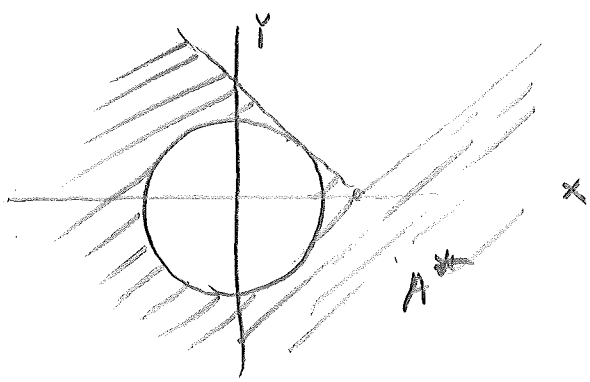


$|Y| \le 2 - |X|$

$A^* = \{(x, y);\ x^2 + y^2 < 2$



A^* IS THE AREA:



THE CIRCLE'S PERIMITER IS INCLUDED IN A^*.

(1-4) MARY

MARY ✓	ARYM	RYMA	YMAR
MRYA	AYMR	RMAY ✓	YARM
MYAR	AMRY ✓	RAYM	YRMA
MAYR	ARMY ✓	RYAM	YMRA
MYRA	AMYR	RAMY ✓	YRAM
MRAY ✓	AYRM	RMYA	YAMR

A TOTAL OF 24 COMBINATIONS $\in A$

A_1 IS	A_2 IS
MARY	MARY
MRAY	MRYA
AMRY	MYAR
ARMY	MAYR
RMAY	MYRA
RAMY	MRAY

$A_1 \cap A_2$ IS

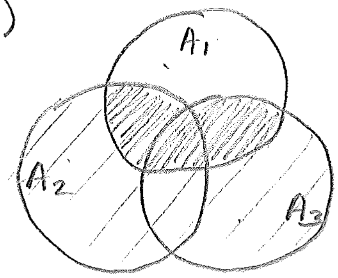
MARY
MRAY

$A_1 \cup A_2$ IS

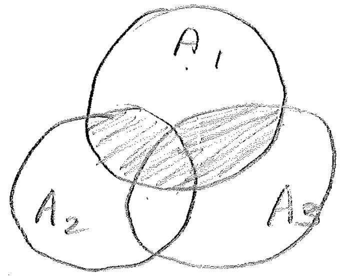
MARY
MRAY
AMRY
ARMY
RMAY
RAMY
MRYA
MYAR
MAYR
MYRA

(1-5)

(a)



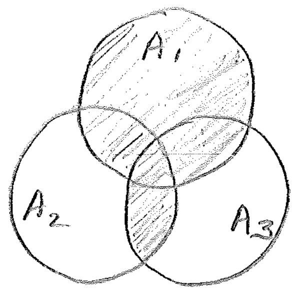
$A_1 \cap (A_2 \cup A_3)$

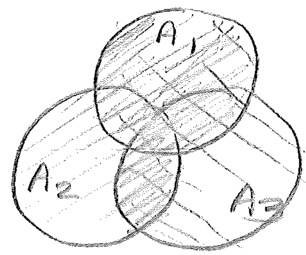


$(A_1 \cap A_2) \cup (A_1 \cap A_3)$

THEY'RE THE SAME

(b)



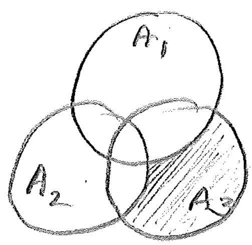


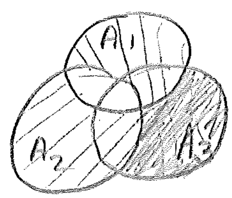
$A_1 \cup (A_2 \cap A_3)$

$(A_1 \cup A_2) \cap (A_1 \cup A_3)$

THEY'RE THE SAME

(c)



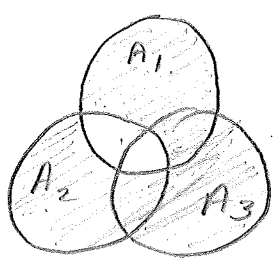


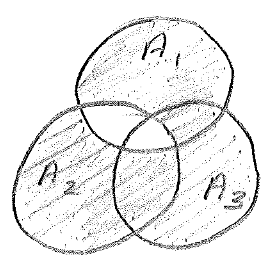
$A_1^* \cap A_2^*$

$(A_1 \cup A_2)^*$

THEY'RE THE SAME

(d)





$(A_1 \cap A_2)^*$

$A_1^* \cup A_2^*$

THEY'RE THE SAME

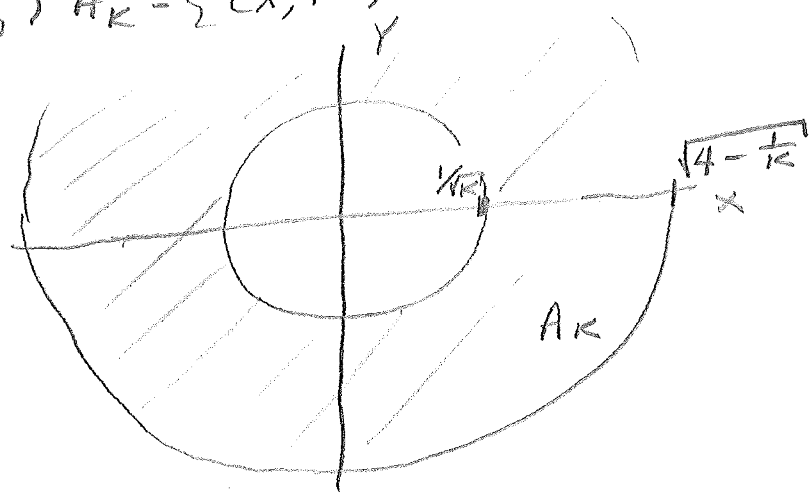
$A_K \subset A_{K+1}$

(1-6) $A_1, A_2, A_3 \cdots$
$\Rightarrow \lim_{K\to\infty} A_K = A_1 \cup A_2 \cup A_3 \cdots$

(a) $A_K = \{x;\ \frac{1}{K} \le x \le 3 - \frac{1}{K}\};\ K = 1, 2, 3, \ldots$

OBVIOUSLY, $\lim_{K\to\infty} A_K = \{x;\ 0 < x < 3\}$

(b) $A_K = \{(x,y);\ \frac{1}{K} \le x^2 + y^2 \le 4 - \frac{1}{K}\};\ K = 1, \ldots$



OBVIOUSLY
$\lim_{K\to\infty} A_K = \{(x,y);\ 0 < x^2 + y^2 < 4\}$

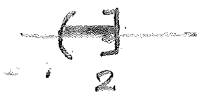
(1-7) $A_1, A_2, A_3 \ldots \Rightarrow A_K \supset A_{K+1}$

$$\Rightarrow \lim_{K\to\infty} A_K = A_1 \cap A_2 \cap A_3 \ldots$$

(a) $A_K = \{x;\ 2 - \frac{1}{K} < x \leq 2\}$; $k = 1, 2, \ldots$

OBVIOUSLY

$$\lim_{K\to\infty} A_K = \{x;\ x = 2\}$$



(b) $A_K = \{x;\ 2 < x \leq 2 + \frac{1}{K}\}$; $K = 1, 2, \ldots$

OBVIOUSLY

$$\lim_{K\to\infty} A_K = \{x;\ x = \phi\}$$

(c) $A_K = \{(x, y);\ 0 \leq x^2 + y^2 \leq \frac{1}{K}\}$; $k = 1, 2, \ldots$

OBVIOUSLY

$$\lim_{K\to\infty} A_K = \{(x, y);\ x = y = 0\}$$

(1-8) $\forall$ 1-D SET A, LET $Q(A) = \sum_A f(x)$

$\Rightarrow f(x) = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^x \quad ; \; x = 0, 1, 2, \ldots$

(a) $A_1 = \{x ; x = 0, 1, 2, 3\}$

$$\Rightarrow Q(A_1) = \frac{2}{3}\left(\frac{1}{3}\right)^0 + \frac{2}{3}\left(\frac{1}{3}\right)^1 + \frac{2}{3}\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3$$

$$= \frac{2}{3}\left[1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}\right]$$

$$= \frac{2}{81}\left[27 + 9 + 3 + 1\right]$$

$$= \frac{2}{81}\left[40\right]$$

$$= 80/81$$

(b) $A_2 = \{x ; x = 0, 1, 2, \ldots\}$

$$Q(A_2) = \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$$

USE GEOMETRIC SERIES:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \forall \; |x| < 1$$

$$\therefore Q(A_2) = \frac{2}{3} \frac{1}{1 - 1/3} = 1.$$

(1-9) $Q(A) = \int_A f(x)\,dx$

$$f(x) = \begin{cases} 6x - 6x^2 & ; \quad 0 < x < 1 \\ 0 & ; \quad \text{ELSEWHERE} \end{cases}$$

(a) $A_1 = \{x ; \frac{1}{4} < x < 3/4\}$

$$\Rightarrow Q(A) = \int_{1/4}^{3/4} [6x - 6x^2]\,dx$$

$$= [3x^2 - 2x^3]_{1/4}^{3/4}$$

$$= \left(3\left(\tfrac{3}{4}\right)^2 - 2\left(\tfrac{3}{4}\right)^3\right) - \left(3\left(\tfrac{1}{4}\right)^2 - 2\left(\tfrac{1}{4}\right)^3\right)$$

$$= \frac{27}{16} - \frac{54}{64} - \frac{3}{16} + \frac{2}{64}$$

$$= \frac{1}{32}[54 - 27 - 6 + 1]$$

$$= \frac{22}{32} = \frac{11}{16}$$

(b) $A_2 = \{x ; x = \frac{1}{2})$

$$\Rightarrow Q(A) = \int_{1/2}^{1/2} f(x)\,dx = 0$$

(c) $A_3 = \{x ; 0 < x < 10\}$

$$Q(A_3) = \int_0^1 [6x^2 - 6x^3]\,dx + \int_1^{10} (0)\,dx$$

$$= 3x^2 - 2x^3 \Big|_0^1$$

$$= 1$$

(1-10) $Q(A) = \#$ POS INT $\subset A$

$A_1 = \{x; x$ IS A MULTIPLE OF $3 \leq 50\}$

$= \{\ldots -3, 0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30,$
$33, 36, 39, 42, 45, 48\}$

$\Rightarrow Q(A_1) = 16$

$A_2 = \{x; x$ IS A MULTIPLE OF $7 \leq 50\}$

$= \{\ldots -7, 0, 7, 14, 21, 28, 35, 42, 49\}$

$\Rightarrow Q(A_2) = 7$

$A_1 \cap A_2 = \{\ldots -21, 0, 21, 42, \ldots\}$

$\Rightarrow Q[A_1 \cap A_2] = 2$

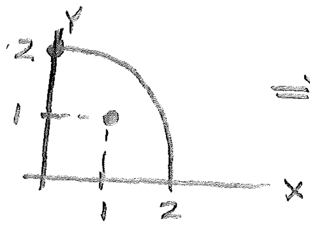
$A_1 \cup A_2 = \{\ldots -7, -3, 0, 3, 6, 7, 9, 12, 14,$
$15, 18, 21, 24, 27, 28, 30, 33, 35, 36, 39, 42, 45, 48, 49, \ldots\}$

$\Rightarrow Q[A_1 \cup A_2] = 21$

$-Q(A_1 \cup A_2) + Q(A_1) + Q(A_2)$
$= 16 + 7 - 21 = 2$

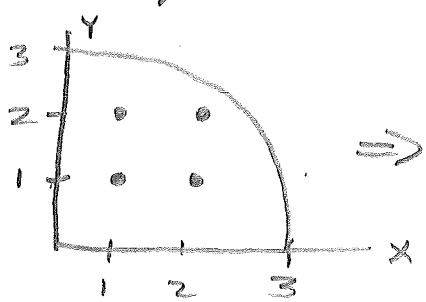
(1-11) $Q(A)$ = # POINTS $(X,Y) \ni$ BOTH X & Y ARE POS. INTEGERS

$A_1 = \{(x,y);\ x^2+y^2 \le 4)$



$\Rightarrow Q(A_1) = 1$

$A_2 = \{(x,y);\ x^2+y^2 \le 9\ )$



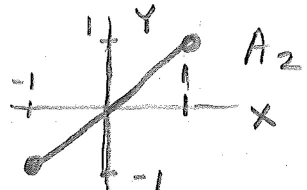
$\Rightarrow Q(A_2) = 4$

(1-12) $Q(A) = \iint_A (x^2 + y^2)\,dx\,dy$

a. $A_1 = \{(x,y);\ -1 \le x \le 1,\ -1 \le y \le 1\}$

$$\Rightarrow Q(A_1) = \int_{-1}^{1}\int_{-1}^{1}(x^2+y^2)\,dx\,dy$$
$$= \int_{-1}^{1}\left[\tfrac{1}{3}x^3 + xy^2\right]_{-1}^{1} dy$$
$$= \int_{-1}^{1}\left[\tfrac{1}{3}\left(1^3 - (-1)^3\right) + (1-(-1))y^2\right]dy$$
$$= \int_{-1}^{1} 2/3 + 2y^2\,dy$$
$$= \tfrac{2}{3}y + \tfrac{2}{3}y^3 \Big|_{-1}^{1}$$
$$= 4/3 + 4/3 = 8/3$$

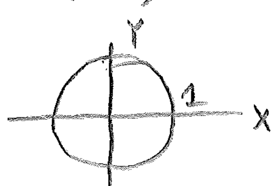
b. $A_2 = \{(x,y);\ -1 \le x = y \le 1)$



BUT $\iint_{1\text{D CURVE}} (x^2+y^2)\,dx\,dy = 0$

$$\Rightarrow Q(A_2) = 0$$

c. $A_3 = \{(x,y);\ x^2+y^2 \le 1)$



LET $r^2 = x^2 + y^2$

$$\Rightarrow \iint_{A_3} (x^2+y^2)\,dx\,dy$$
$$= \int_0^{2\pi} d\theta \int_0^1 r\,r^2\,dr$$
$$= 2\pi \quad \tfrac{1}{4}r^4\Big|_0^1$$
$$= \pi/2$$

(1-13) $Q(A) = \iiint_A z \, dx\,dy\,dz$

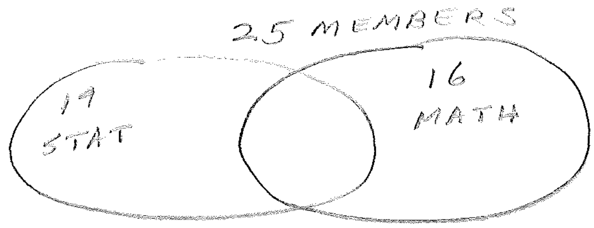
$A_1 = \{(x,y,z);\ (x,y,z) = (0,0,0)\}$

$Q(A_1) = 0$

$A_2 = \{(x,y,z);\ x^2 + y^2 \leq 4\ ;\ 0 \leq z < 1$

$$Q(A) = \iint \int_0^1 z \, dz\,dx\,dy$$
$$= \iint \tfrac{1}{2} z^2 \Big|_0^1 dx\,dy$$
$$= \tfrac{1}{2} \underbrace{\iint dx\,dy}_{\text{CIRCLE OF RADIUS } 2}$$
$$= \tfrac{1}{2}(\pi 4)$$
$$= 2\pi$$

F14



STATS $\cap$ MATH = MATH STATS

MATH $\cup$ STATS = 25

STATS = 19

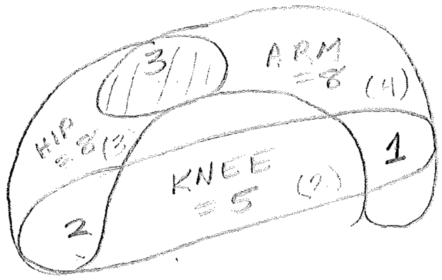
MATH = 16

STATS $\cap$ MATH = MATH + STATS − MATH $\cup$ STATS

= 16 + 19 − 25

= 35 − 25 = 10

1-15



WE CAN BREAK THIS INTO DISJOINT GROUPS:

ONLY HIP = 3
" ARM = 4
" KNEE = 2
HIP & ARM = 3
ARM & KNEE = 1
KNEE & HIP = 2
ALL 3 = 0

15 PLAYERS

BUT WE ONLY HAD 11 PLAYERS
SO THE REPORT IS WRONG

1-16 $Q(A) = \int_A x^2 dx$

(a) $A_k = \{x ; 0 \le x \le 2 - \frac{1}{k}\}$; $k = 1, 2, 3, \ldots$

$\lim_{k \to \infty} A_k = \{x ; 0 \le x < 2\}$

$$\Rightarrow Q(\lim_{k \to \infty} A_k) = \int_0^2 x^2 dx$$
$$= \frac{1}{3} x^3 \Big|_0^2 = \frac{8}{3}$$

$$Q(A_k) = \int_0^{2 - \frac{1}{k}} x^2 dx$$
$$= \frac{1}{3} x^3 \Big|_0^{2 - \frac{1}{k}}$$
$$= \frac{1}{3} \left(2 - \frac{1}{k}\right)^3$$

$$\lim_{k \to \infty} Q(A_k) = 8/3$$

(b) $A_k = \{x ; |x| \le 1 + \frac{1}{k}\}$ $\quad k = 1, 2, 3, \ldots$

$\lim_{k \to \infty} A_k = \{x ; |x| \le 1\}$

$$Q[\lim_{k \to \infty} A_k] = \int_{-1}^{1} x^2 dx$$
$$= 2 \int_0^1 x^2 dx$$
$$= \frac{2}{3} x^3 \Big|_0^1 = \frac{2}{3}$$

$$Q(A_k) = \int_{-(1+\frac{1}{k})}^{(1+\frac{1}{k})} x^2 dx$$

$$= 2 \int_0^{(1+\frac{1}{k})} x^2 dx$$
$$= \frac{2}{3} x^3 \Big|_0^{1+\frac{1}{k}}$$
$$= \frac{2}{3} \left(1 + \frac{1}{k}\right)^3$$

$$\therefore \lim_{k \to \infty} Q(A_k) = \frac{2}{3}$$

(1-17) $\mathcal{C} = \{c;\ c = 1, 2, 3, 4, 5, 6\}$

$C_1 = \{c;\ c = 1, 2, 3, 4\}$

$C_2 = \{c;\ c = 3, 4, 5, 6\}$

$P[C_1] = 4/6 = 2/3$

$P[C_2] = 4/6 = 2/3$

$C_1 \cap C_2 = \{x;\ x = 3, 4\}$

$\Rightarrow P[C_1 \cap C_2] = 2/6 = \frac{1}{3}$

$C_1 \cup C_2 = \{x;\ x = 1, 2, 3, 4, 5, 6\}$

$\Rightarrow P[C_1 \cup C_2] = P[\mathcal{C}] = \underline{1}$

(1-18) $C_1 = \{c;\ c = AH, 2H, \ldots, QH, KH\}$

$P[C_1] = 13/52 = 1/4$

$C_2 = \{c;\ c = KH, KS, KD, KC\}$

$P[C_2] = 4/52 = 2/26 = 1/13$

$C_1 \cap C_2 = \{c;\ c = KH\}$

$\Rightarrow P[C_1 \cap C_2] = 1/52$

$C_1 \cup C_2 = \{c;\ c = AH, 2H, 3H, 4H, 5H, 6H, 7H, 8H, 9H, 10H, JH, QH, KH, KS, KC, KD\}$

$\Rightarrow P[C_1 \cup C_2] = 16/52$

$= \frac{4}{13}$

(1-19) $P(C_i) = \frac{1}{2^i}$; $i = 1, 2, \ldots,$

$$P(\mathcal{C}) = \sum_{i=1}^{\infty} \frac{1}{2^i} = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i$$

USE GEOMETRIC SERIES

$$\sum_{i=0}^{\infty} X^i = \frac{1}{1-x} \quad ; \quad |x| < 1$$

NOW

$$\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i - \left(\frac{1}{2}\right)^0 = \frac{1}{1-1/2} - 1 = \underline{1}$$

$C_1 = \{c;\ c = H, TH, TTH, TTTH, TTTTH)$

$$P[C_1] = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{1}{32}[16 + 8 + 4 + 2 + 1] = \frac{31}{32}$$

$C_2 = \{c;\ c = TTTTH, TTTTTH\}$

$$\Rightarrow P[C_2] = \frac{1}{2^5} + \frac{1}{2^6} = \frac{1}{32} + \frac{1}{64} = 3/64$$

$C_1 \cap C_2 = \{c;\ c = TTTTH\}$

$$= 1/32$$

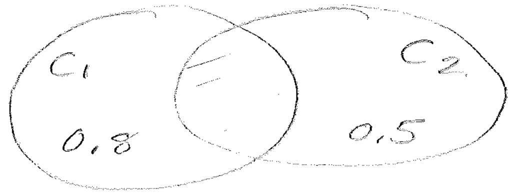
$C_1 \cup C_2 = \{c;\ c = H, TH, TTH, TTTH,$ $TTTTH, TTTTTH\}$

$$= \frac{31}{32} + \frac{1}{64} = \frac{63}{64}$$

(1-20) $\mathcal{E} = C_1 \cup C_2$

$P(C_1) = 0.8$

$P(C_2) = 0.5$



$P[C_1 \cup C_2] = 1$

$= P(C_1) + P(C_2) - P(C_1 \cap C_2)$

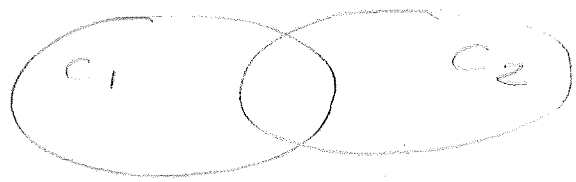
OR

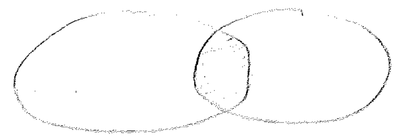
$P(C_1 \cap C_2) = P(C_1) + P(C_2) - 1$

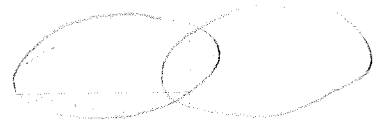
$= 1.3 - 1$

$= 0.3$

(1-21) WE CAN SHOW THIS WITH A VENN DIAGRAM:



$P(C_1 \cap C_2) \Rightarrow$ 

$P(C_1) \Rightarrow$ 

SINCE $C_1 \cap C_2 \subset C_1$, IT FOLLOWS THAT $P(C_1 \cap C_2) \leq P(C_1)$

SIMILARLY,

$$C_1 \cup C_2 \subset C_1$$

$$\Rightarrow P[C_1 \cup C_2] \leq P[C_1]$$

ALSO.

$$P(C_1) + P(C_2) - P[C_1 \cap C_2] = P(C_1 \cup C_2)$$

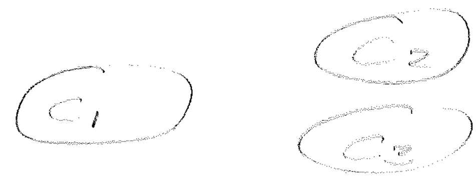
SINCE $P[C_1 \cap C_2] > 0$, IT FOLLOWS THAT

$$P(C_1) + P(C_2) \leq P(C_1 \cup C_2)$$

IN SUMMARY

$$P(C_1 \cap C_2) \leq P(C_1) \leq P(C_1 \cup C_2) \leq P(C_1) + P(C_2)$$

(1-22) SINCE $C_1, C_2, \& C_3$ ARE DISJOINT, WE MAY REPRESENT AS:



OBVIOUSLY:

$$(C_1 \cup C_2) \cap C_3 = 0$$

$$\Rightarrow P[(C_1 \cap C_2) \cap C_3] = 0$$

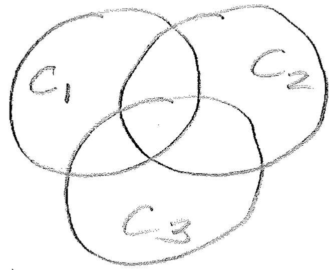
CONSIDER, THEN, $C_1^* \cup C_2^*$

OBVIOUSLY, $C_1^* \cup C_2^* = \mathscr{C}$

AND

$$P[\mathscr{C}] = 1$$

(1-23) LET'S LOOK @ IT WITH A VENN DIAGRAM:



OBVIOUSLY

$$P[C_1 \cup C_2 \cup C_3] = P[C_1] + P[C_2] + P[C_3] - P[C_1 \cap C_2] - P[C_1 \cap C_3] - P[C_2 \cap C_3] + P[C_1 \cap C_2 \cap C_3]$$

IN GENERAL

$$P\left[\bigcup_{k=1}^{n} C_k\right] = \sum_{k=1}^{\binom{n}{1}} P[C_k]$$

$$- \sum_{k=1}^{\binom{n}{2}} P[\text{all } C_i\text{'s} \overset{\text{INTER}}{\cap} 2 \text{ AT A TIME}]$$

$$+ \sum_{k=1}^{\binom{n}{3}} P[\text{ALL } C_i\text{'s} \overset{\text{INTER}}{\cap} 3 \text{ AT A TIME}]$$

$$- \cdots$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{\binom{n}{i}} P[\text{ALL } C\text{'s INTERSECTED } i \text{ AT A TIME}]$$

(1-24) $\mathscr{C} = \{c;\ c \text{ IS A PLAYING CARD}\}$

$$X(c) = \begin{cases} 4 & , c \text{ IS AN ACE} \\ 3 & c \text{ " A KING} \\ 2 & c \text{ " A QUEEN} \\ 1 & c \text{ " " JACK} \\ 0 & \text{OTHERWISE} \end{cases}$$

$P(c) = 1/52 \quad \forall c$

$\mathscr{A} = \{x;\ x = 0, 1, 2, 3, 4\}$

$P[c = \text{ACE}] = P[X(c) = 4] = 1/13$

$P[c = \text{KING}] = P[X(c) = 3] = 1/13$

$P[c = \text{QU}] = P[X(c) = 2] = 1/13$

$P[c = \text{JACK}] = P[X(c) = 1] = 1/13$

$P[X(c) = 0] = 9/13$

(1-25) $\mathscr{C} = \{c;\ 0 < c < 10\}$

$C \subset \mathscr{C}$

$P[C] = \int_C \frac{1}{10}\, dz$

$X = X(c) = 2c - 10$

$\therefore \mathscr{A} = \{x;\ -10 < x < 10\}$

$A \subset \mathscr{A}$

TO HAVE $P[C] = P_X(A)$

$P_X(A) = \frac{1}{20} \int_A dx$

OR

$\mathscr{C} = \{c;\ 0 < c < 10$

$P(C) = \int_C \frac{1}{10}\, dx \ ,\ c \in \mathscr{C}$

Let $x = 2c - 10$

$\Rightarrow \mathscr{A}\ \{x;\ -10 < a < 10\}$

$P(A) = P(C) = \int_C \frac{1}{10}\, dz$

LET $x = 2z - 10$

$\Rightarrow P(A) = \int_A \frac{1}{20}\, dx$

(1-26) $P(A) = \sum_A \sum f(x, y)$

$f(x, y) = \frac{1}{52}$

$(x, y) \in A = \{(x, y) : (x; y) = [(0, 1), (0, 2)$

$\ldots (0, 13), (1, 1), \ldots, (1, 13), \ldots (3, 13)]$

FIND $P(A) = Pr[(X, Y) \in A]$

(a) $\ni A = \{(x, y) ; (x, y) = (0, 4), (1, 3), (2, 2)\}$

$P(A) = 3/52$

(b) $\ni A = \{(x, y) ;\ x + y = 4 \text{ ∻ } x + y \in A\}$

$= \{(0, 4), (1, 3), (2, 2), (3, 1)\}$

$= 4/52 = 1/13$

(1-27) $P(A) = \int_A f(x)\,dx$

$f(x) = 2x/9$

$x \in A = \{x;\ 0 < x < 3\}$

$A_1 = \{x;\ 0 < x < 1\}$

$A_2 = \{x;\ 2 < x < 3\}$

$P(A_1) = \int_0^1 \frac{2x}{9}\,dx$

$= \frac{2}{9}\ \frac{1}{2}x^2\Big|_0^1 = \frac{1}{9}$

$P(A_2) = \frac{2}{9}\int_2^3 x\,dx$

$= \frac{1}{9}\ x^2\Big|_2^3$

$= \frac{1}{9}[9-4] = 5/9$

$P[A_1 \cup A_2] = P[A_1] + P[A_2] = 6/9$

(1-28) $A = \{x;\ 0 < x < 1\}$

$A_1 = \{x;\ 0 < x < \frac{1}{2}\}$

$A_2 = \{x;\ \frac{1}{2} \leq x < 1\}$

SINCE $A = A_1 \cup A_2$ & $A_1 \cap A_2 = \phi$

THEN

$P(A) = P(A_1) + P(A_2) = 1$

GIVEN $P(A_1) = 1/4$

$\Longrightarrow P(A_2) = 3/4$

(1-29) $\mathcal{A} = \{x ; 0 < x < 10\}$

$P(A_1) = 3/8$

$\Rightarrow A_1 = \{x ; 1 < x < 5\}$

$A_2 = \{x ; 5 \le x < 10\}$

OBVIOUSLY

$A_1 \cup A_2 \subset \mathcal{A}$ AND $A_1 \cap A_2 = \phi$

$\Rightarrow P(A_1 \cup A_2) = P(A_1) + P(A_2) \le P(\mathcal{A}) = \underline{1}$

GIVEN $P(A_1) = 3/8$

$P(A_2) \le 1 - P(A_1) = 5/8$

(1-30) $A = \{x;\ 0 < x < 1\}$

$A_1 = \{x;\ \frac{1}{4} < x < \frac{1}{2}\}$

$P(A_1) = \frac{1}{8}$

$A_2 = \{x;\ \frac{1}{2} \le x < 1\}$

$P(A_2) = \frac{1}{2}$

SINCE $A_1 \cap A_2 = \phi$;

$P[A_1 \cup A_2] = P[A_1] + P(A_2) = 5/8$

NOW

$P(A_1^*) = 1 - P(A_1) = 5/8$

$P(A_1^* \cap A_2^*) = P(A_3)$

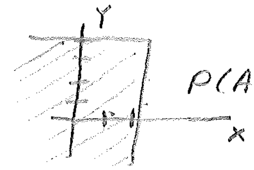
$\Rightarrow A_3 = \{x;\ 0 < x \le \frac{1}{4})$

SINCE A_1 A_2 & A_3 ARE DISJOINT

$P(A_3) = 1 - P(A_2) - P(A_1) = 2/8$

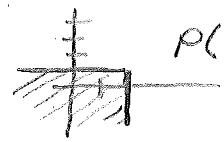
(1-31)

$A_1 = \{(x,y);\ x \le 2,\ y \le 4\}$



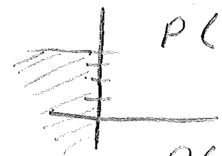
$P(A_1) = 7/8$

$A_2 = \{(x,y);\ x \le 2,\ y \le 1\}$



$P(A_2) = 4/8$

$A_3 = \{(x,y);\ x \le 0,\ y \le 4\}$



$P(A_3) = 3/8$

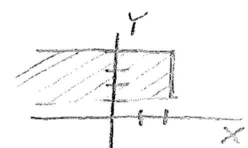
$A_4 = \{(x,y);\ x \le 0,\ y \le 1\}$

$P(A_4) = 2/8$

SINCE $A_2 \subset A_1$

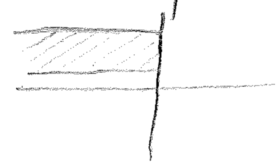
$P((A_1 \cap A_2^*)) = P(A_1) - P(A_2) = \frac{3}{8}$

$A_1 \cap A_2^* = \{(x;y);\ x < 2,\ 1 < y \le 4\}$



SINCE $A_4 \subset A_3$

$P(A_2 \cap A_3^*) = 1/8$



THEN

$A_5 = (A_1 \cap A_2^*) \cap (A_2 \cap A_3)^*$

$= 3/8 - 1/8 = 1/4$

(1-32) $P(A) = \int_A e^{-x}dx$

$A = \{x;\ 0 < x < \infty\}$

$A_K = \{x;\ 2 - \frac{1}{K} < x \leq 3\}$

$\Rightarrow P(A_K) = \int_{2-\frac{1}{K}}^{3} e^{-x}dx$

$= e^{-x}\Big|_{3}^{2-\frac{1}{K}}$

$= e^{\frac{1}{K}-2} - e^{-3}$

$\lim\limits_{K\to\infty} P(A_K) = e^{-2} - e^{-3}$

$\lim\limits_{K\to\infty} A_K = \{x;\ 2 \leq x \leq 3\}$

$P[\lim\limits_{K\to\infty} A_K] = \int_2^3 e^{-x}dx = e^{-2} - e^{-3}$

AND

$\lim\limits_{K\to\infty} P(A_K) = P(\lim\limits_{K\to\infty} A_K)$

(1-33) (a) $f(x) = C\left(\frac{2}{3}\right)^x$; $x = 1, 2, 3, \ldots$

$$1 = C \sum_{x=1}^{\infty} \left(\frac{2}{3}\right)^x$$

$$= C\left[\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n - 1\right]$$

USING GEOMETRIC SERIES:

$$1 = C\left[\frac{1}{1-2/3} - 1\right]$$

$$= 2C \Rightarrow C = 1/2$$

(b) $f(x) = \begin{cases} c\,x\,e^{-x} & ; \ 0 < x < \infty \\ 0 & ; \ \text{ELSEWHERE} \end{cases}$

$$1 = C\int_0^{\infty} x e^{-x}\,dx$$

$$u = x \qquad dv = e^{-x}dx$$

$$du = dx \qquad v = -e^{-x}$$

$$\Rightarrow 1 = C\left[-x e^{-x}\Big|_0^{\infty} + \int_0^{\infty} e^{-x}dx\right]$$

$$= C\left[-(0-0) - e^{-x}\Big|_0^{\infty}\right]$$

$$= C\left[0 - 1\right] = C$$

$$\Rightarrow C = 1$$

(1-34) $f(x) = \begin{cases} \frac{x}{15} & ; x = 1, 2, 3, 4 5 \\ 0 & ; \text{ELSEWHERE} \end{cases}$

$Pr[X = 1 \text{ OR } 2] = \frac{1}{15} + \frac{2}{15} = \frac{3}{15}$

$Pr[\frac{1}{2} < X < \frac{5}{2}] = Pr[X = 1 \text{ OR } 2]$

$= 3/15$

$Pr[1 \leq X \leq 2] = Pr[X = 1 \text{ OR } 2]$

$= 3/15$

(1-35) $f(x) = \begin{cases} \frac{1}{x^2} & ; \ 1 < x < \infty \\ 0 & ; \ \text{ELSEWHERE} \end{cases}$

$A_1 = \{x ; 1 < x < 2\}$

$A_2 = \{x ; 4 < x < 5\}$

$A_1 \cap A_2 = \{\phi\}$

$\Rightarrow Pr[A_1 \cap A_2] = 0$

$A_1 \cup A_2 = \{x ; 1 < x < 2 \text{ OR } 4 < x < 5\}$

$\Rightarrow Pr[A_1 \cup A_2] = \int_1^2 \frac{1}{x^2} dx + \int_4^5 1/x^2 dx$

$= +\frac{1}{x}\Big|_2^1 + \frac{1}{x}\Big|_5^4$

$= (1 - \frac{1}{2}) + (\frac{1}{4} - \frac{1}{5})$

$= \frac{1}{2} + \frac{1}{20}$

$= 11/20$

(1-36) $f(x_1, x_2) = \begin{cases} 4x_1x_2 & ; 0<x_1<1,\ 0<x_2<1 \\ 0 & ; \text{ELSEWHERE.} \end{cases}$

a. $Pr[0 < X_1 < \frac{1}{2},\ \frac{1}{4} < X_2 < 1]$

$= 4\int_0^{\frac{1}{2}} \int_{1/4}^{1} x_1 x_2 \, dx_1 dx_2$

$= 4\int_0^{1/2} x_1 \ \frac{1}{2} x_2^2 \Big|_{1/4}^{1}$

$= 4 \cdot [1 - \frac{1}{16}] \int_0^{1/2} x_1 dx_1$

$= 2 [1 - \frac{1}{16}] \ \frac{1}{2} x_1^2 \Big|_0^{1/2}$

$= \frac{30}{16} \ \frac{1}{4} = \frac{15}{64}$

$= \frac{15}{16} \ \frac{1}{4} = \frac{15}{64}$

b. $Pr[X_1 = X_2] = 0$

c. $Pr[X_1 < X_2] = 4\int_0^1 \int_0^{x_2} x_1 x_2 \, dx_1 dx_2$

$= 4\int_0^1 x_2 \ \frac{1}{2} x_1^2 \Big|_0^{x_2} dx_2$

$= 2\int_0^1 x_2^3 dx_2 = 2/3$

$= \frac{2}{3} X_2^3 \Big|_0^1 = 2/3$

d. $Pr[X_1 \leq X_2] = Pr[X_1 < X_2] = 2/3$

(1-37) $f(x_1 x_2 x_3) = \begin{cases} e^{-(x_1+x_2+x_3)} & ; x_1, x_2, x_3 > 0 \\ 0 & ; \text{OTHERWISE} \end{cases}$

(a) $Pr[X_1 < X_2 \leq X_3]$

$$= \int_0^\infty \int_0^{x_3} \int_0^{x_2} e^{-x_1} dx_1 \, e^{-x_2} dx_2 \, e^{-x_3} dx_3$$

$$= \int_0^\infty \int_0^{x_3} e^{-x_1}\Big|_{x_2}^{0} e^{-x_2} dx_2 \, e^{-x_3} dx_3$$

$$= \int_0^\infty \int_0^{x_3} (1 - e^{-x_2}) e^{-x_2} dx_2 \, e^{-x_3} dx_3$$

$$= \int_0^\infty \int_0^{x_3} [e^{-x_2} - e^{-2x_2}] dx_2 \, e^{-x_3} dx_3$$

$$= \int_0^\infty [e^{-x_2}\Big|_{x_3}^{0} + \tfrac{1}{2} e^{-2x_2}\Big|_0^{x_3}] e^{-x_3} dx_3$$

$$= \int_0^\infty [1 - e^{-x_3} + \tfrac{1}{2} e^{-2x_3} - \tfrac{1}{2}] e^{-x_3} dx_3$$

$$= \int_0^\infty \tfrac{1}{2} e^{-x_3} - e^{-2x_3} + \tfrac{1}{2} e^{-3x_3}] dx_3$$

$$= \tfrac{1}{2} e^{-x_3}\Big|_\infty^0 + \tfrac{1}{2} e^{-2x_3}\Big|_0^\infty + \tfrac{1}{6} e^{-3x_3}\Big|_\infty^0$$

$$= \tfrac{1}{2}[1-0] + \tfrac{1}{2}[0-1] + \tfrac{1}{6} = \tfrac{1}{6}$$

(b) $Pr[X_1 = X_2 < X_3] = 0$

(1-38) MODE = $X_M \ni f(X_M) > f(X) \;\forall\; X \neq X_M$

(a) $f(x) = (\frac{1}{2})^x$; $x = 1, 2, 3, \ldots$

$X_m = \frac{1}{2}$

(b) $f(x) = \begin{cases} 12x^2(1-x) & ; 0 < x < 1 \\ 0 & ; \text{ELSEWHERE} \end{cases}$

$$\frac{d}{dX} 12X^2(1-X) = \frac{d}{dX} 12X^2 - 12X^3$$
$$= 12[2X - 3X^2] = 0$$
$$\Rightarrow 2 = 3X \Rightarrow X = 2/3$$
$$\therefore X_m = 2/3$$

(c) $f(x) = \begin{cases} \frac{1}{2}x^2 e^{-x} & ; 0 < x < \infty \\ 0 & ; \text{ELSEWHERE} \end{cases}$

$$\frac{1}{2}\frac{d}{dx} x^2 e^{-x} = \frac{1}{2}[2xe^{-x} - x^2 e^{-x}]$$
$$= 0 \Rightarrow 2x = x^2$$
$$\Rightarrow x_M = 2$$

(1-39) MEDIAN $= X_m \ni$

$$P_r[X < x] \le \tfrac{1}{2} \;,\; Pr[X \le x] \ge \tfrac{1}{2}$$

(a) $f(x) = \dfrac{4!}{x!\,(4-x)!} \left(\tfrac{1}{4}\right)^x \left(\tfrac{3}{4}\right)^{4-x} \;;\; x = 0, 1, 2, 3, 4$

$$P[X = 0] = \frac{4!}{0!\,4!} \left(\tfrac{1}{4}\right)^0 \left(\tfrac{3}{4}\right)^4 = \frac{81}{256}$$

$$P[X = 1] = \frac{4!}{1!\;3!} \left(\tfrac{1}{4}\right)^1 \left(\tfrac{3}{4}\right)^3 = \frac{108}{256}$$

$$\begin{array}{r} 27 \\ 4 \\ \hline 108 \end{array}$$

$\Rightarrow \quad X_m = \underline{1}$

$$P_r[X < 1] = P_r[X = 0] = \frac{81}{256} \le \frac{1}{2}$$

$$P_r[X \le 1] = P_r[X = 1] + P_r[X = 0] = {}^{189}/_{256} \ge \frac{1}{2}$$

(b) $f(x) = \begin{cases} 3x^2 \;; & 0 < x < 1 \\ 0 \;; & \text{ELSEWHERE} \end{cases}$

$$3\int_0^{x_m} x^2\,dx = 3\int_{x_m}^{1} x^2\,dx = 1/2$$

$$\therefore \tfrac{1}{2} = x^3 \Big|_0^{x_m} = x_m^3 \Rightarrow x_m = \sqrt[3]{1/2}$$

~~(c) $f(x) = \begin{cases} \tfrac{1}{2} x^2 e^{-x} \;; & 0 < x < \infty \\ 0 & \text{ELSEWHERE} \end{cases}$~~

~~$\tfrac{1}{2}\int_0^{x_m} x^2 e^{-x}\,dx =$~~

(c) $f(x) = \dfrac{1}{\pi(1 + x^2)}$

$$f(x) = f(-x) \Rightarrow x_m = 0$$

(1-40) $0 < p < 1$

$$f(x) = \begin{cases} 4x^3 & ; 0 \le X < 1 \\ 0 & ; \text{ELSEWHERE} \end{cases}$$

FIND $\xi_p \ni$

$$P[X < \xi_p] \le p = .20$$

$$P[X \le \xi_p] \ge p = .20$$

$$p = \int_0^{\xi_p} 4X^3 dx = X^4 \Big|_0^{\xi_p} = \xi_p^4 \Rightarrow \xi_p = \sqrt[4]{.2}$$

(1-41) (a) $f(x) = \begin{cases} 12x(1-x)^2 & ; 0 \le x < 1 \\ 0 & ; \text{OTHERWISE} \end{cases}$

$$\frac{d}{dx} f(x) = 0 = (1-x)^2 + x \frac{d}{dx}(x^2 - 2x - 1)$$

$$= (x^2 + 2x - 1) + x(2x - 2)$$

$$= x^2 + \cancel{2x} - 1 + 2x^2 - \cancel{2x}$$

$$= 3x^2 - 1 \Rightarrow x = \sqrt{\tfrac{1}{3}}$$

(b) LET $I = b - a$

THUS, CONSIDER

$$\int_a^{a+I} f(x)\,dx$$

$$\cancel{\frac{d}{da} \int_a^{a+I} f(x)\,dx = f(I+a) - f(a)}$$

NOT SO GOOD. LETS LOOK @ HINT

$$\int_a^b f(x)\,dx = p = \int_a^{a+I} f(x)\,dx$$

CONSIDER $I(a)$

AND $\frac{d}{da} I = 0$

NOW

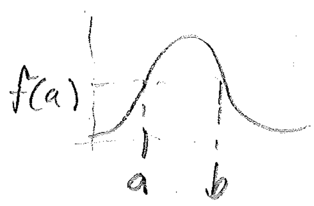
$$\frac{d}{dx} \int_{v(x)}^{u(x)} f(x,y)\,dy = \int_{v(x)}^{u(x)} \frac{d}{dx} f(x,y)\,dy + \frac{du(x)}{dx} f(u(x)) - \frac{dv(x)}{dx} f[v(x)]$$

THUS

$$\frac{d}{da} \int_a^{a+I} f(x)\,dx = \left(1 + \frac{dI}{da}\right) f(a) - f(b) = \frac{dp}{da} = 0$$

BUT, SINCE $\frac{dI}{da} = 0$

$$f(a) = f(b)$$



THUS, $a < x_m < b$

$$k(a,b) = b - a - \lambda\left[\int_a^b f(x)\,dx - p\right]$$

(1-42) $\int_0^\infty g(x)\,dx = 1$

$$f(x_1 x_1) = \frac{2 g\left(\sqrt{x_1^2 + x_2^2}\right)}{\pi \sqrt{x_1^2 + x_2^2}}$$

$$\iint f(x_1 x_2)\,dx_1 dx_2 = \frac{2}{\pi} \int_0^\infty \int_0^\infty \frac{g\left(\sqrt{x_1^2 + x_2^2}\right)}{\left(\sqrt{x_1^2 + x_2^2}\right)} dx_1 dx_2$$

GO TO POLAR COORD

$$dx_1 dx_2 = r\,dr\,d\phi$$

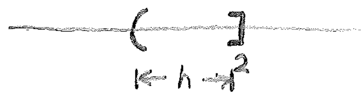
$$\iint f(x_1 x_2)\,dx_1 dx_2 = \frac{2}{\pi} \int_0^\infty \int_0^{\pi/2} \frac{g(r)}{r}\, r\,dr\,d\phi$$

$= 1$

(1-43) $f(x) = \frac{x}{10}$, $x = 1, 2, 3, 4$

(a) $h > 0$

$Pr[2 - h < X \le 2]$



IF $h < 1$,

$Pr[2 - h < X \le 2] = 2/10$

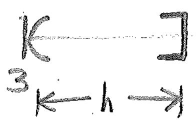
$\lim_{h \to 0} Pr[2 - h < X \le 2] = 2/10$

NOW

$\lim_{h \to 0} [2 - h < X \le 2] \Rightarrow 2 = X$

$\Rightarrow Pr[\lim_{h \to 0} 2 - h < X \le 2] = 2/10$

(b) $Pr[3 < X \le 3 + h]$



FOR $h < 1$

$Pr[3 < X \le 3 + h] = 0$

$\lim_{h \to 0} Pr[3 < X \le 3 + h] = 0$

NOW

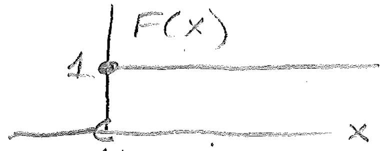
$\lim_{h \to 0} [3 < X \le 3 + h]$

IS SUCH THAT $X \ne 3$

$\Rightarrow Pr[\lim_{h \to 0} 3 < X \le 3 + h] = 0$

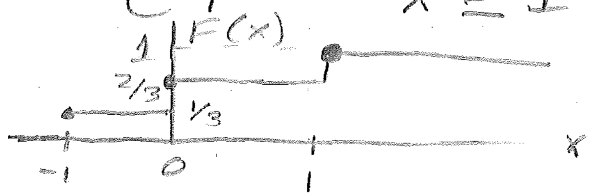
(1-44) (a) $f(x) = \begin{cases} 1 & ; x = 0 \\ 0 & ; \text{ELSEWHERE} \end{cases}$

$F(x) = \begin{cases} 1 & ; x \geq 0 \\ 0 & x < 0 \end{cases}$



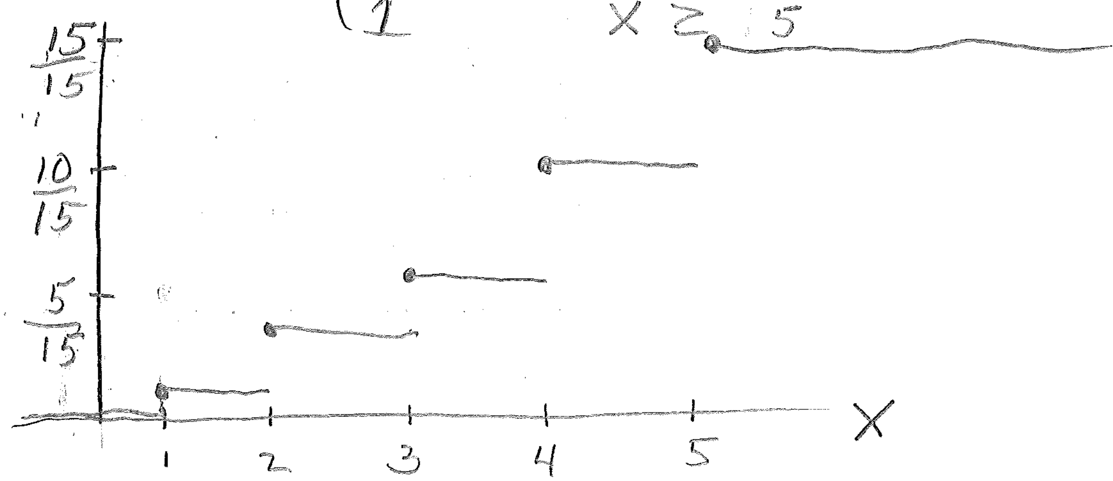
(b) $f(x) = \begin{cases} \frac{1}{3} & ; x = -1, 0, 1 \\ 0 & ; \text{ELSEWHERE} \end{cases}$

$F(x) = \begin{cases} 0 & ; x < -1 \\ \frac{1}{3} & -1 \leq x < 0 \\ \frac{2}{3} & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$



(c) $f(x) = \begin{cases} \frac{x}{15} & ; x = 1, 2, 3, 4, 5 \\ 0 & ; \text{ELSEWHERE} \end{cases}$

$F(x) = \begin{cases} 0 & x < 1 \\ 1/15 & 1 \leq x < 2 \\ 3/15 & 2 \leq x < 3 \\ 6/15 & 3 \leq x < 4 \\ 10/15 & 4 \leq x < 5 \\ 1 & x \geq 5 \end{cases}$

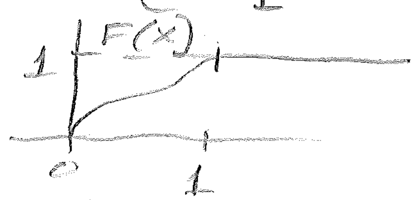


(d) $f(x) = \begin{cases} 3(1-2x+x^2) & ; 0 < x < 1 \\ 0 & ; \text{ELSEWHERE} \end{cases}$

$3\int_0^x (1-2x+x^2)\,dx$

$= 3\left[x - x^2 + \frac{1}{3}x^3\right]$

$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ 3(x - x^2 + \frac{1}{3}x^3) & \\ \quad = 3x(1 - x + \frac{1}{3}x^2) & ; 0 \le x < 1 \\ 1 & ; x \ge 1 \end{cases}$

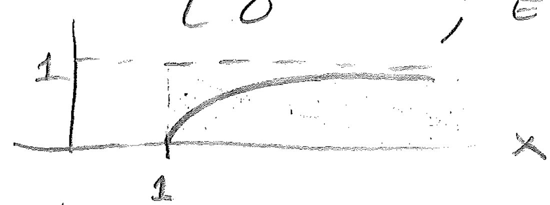


e. $f(x) = \begin{cases} \frac{1}{x^2} & ; 1 < x < \infty \\ 0 & \text{ELSEWHERE} \end{cases}$

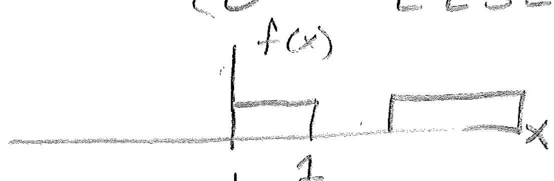
$\int_1^x x^{-2}\,dx = -x^{-1}\Big|_1^x = -\frac{1}{x}\Big|_1^x$

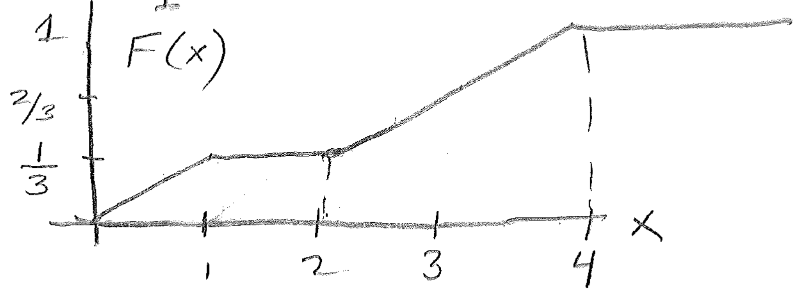
$= 1 - \frac{1}{x}$

$F(x) = \begin{cases} 1 - \frac{1}{x} & ; x \ge 1 \\ 0 & ; \text{ELSEWHERE} \end{cases}$



f. $f(x) = \begin{cases} \frac{1}{3} & ; 0 < x < 1 \text{ OR } 2 < x < 4 \\ 0 & \text{ELSEWHERE} \end{cases}$





(1-45) X_m = MEDIAN

$P[X < x] \leq \frac{1}{2}$ $P[X \leq x] \geq 1/2$

(a) $X_m = 0$

(b) $X_m = 0$

(c) $X_m = 4$

(e) $X_m = 2$

(f) $X_m = 5/2$

FOR (d):

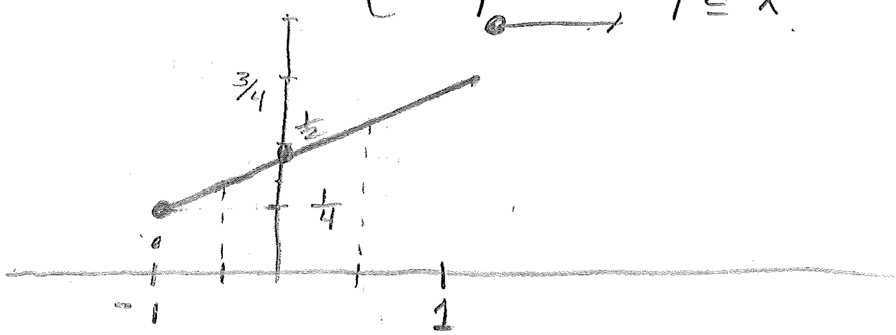
$$F(x) = 3x(1 - x + \tfrac{1}{3}x^2) \quad ; \quad 0 \leq x \leq 1$$

$$= x(x^2 - 3x + 3)$$

DEFINE $X_m \ni X_m(X_m^2 - 3X_m + 3) = \frac{1}{2}$

ENVOLVES SOLN' OF THIS TRANS EQ

(1-46) $F(x) = \begin{cases} 0 & ; x < -1 \\ \frac{x+2}{4} & ; -1 \le 1 < 1 \\ 1 & ; 1 \le x \end{cases}$



(a) $P_r[-\frac{1}{2} < X \le \frac{1}{2}] = \frac{5}{8} - \frac{3}{8} = \frac{1}{4}$

(b) $P_r[X = 0] = 0$

(c) $P_r[X = 1] = 1 - \frac{3}{4} = \frac{1}{4}$

(d) $P_r[2 < X \le 3] = 1 - 1 = 0$

(1-47) $F(x,Y)$

$$f(x,Y) = \frac{\delta^2 F(x,Y)}{\delta x \delta Y}$$

$$P_r[a < x \leq b,\ c < Y \leq d]$$

$$= \int_a^b \int_c^d \frac{\delta^2 F(x,Y)}{\delta x \delta Y}\, dY\, dx$$

$$= \int_a^b \left. \frac{\delta F(x,Y)}{\delta x} \right|_c^d dx$$

$$= \int_a^b \left[\frac{\delta F(x,d)}{\delta x} - \frac{\delta F(x,c)}{\delta x} \right] dx$$

$$= F(x,d)\Big|_a^b - F(x,c)\Big|_a^b$$

$$= F(b,d) - F(a,d) - F(b,c) + F(a,c)$$

(1-48) $f(x) = \begin{cases} 1 & ; 0 < x < 1 \\ 0 & ; \text{ELSEWHERE} \end{cases}$

$Y = \sqrt{X}$

$F(Y) = P[Y \leq Y]$

$= P[\sqrt{X} \leq Y]$

$= P[X \leq \sqrt{Y}]$

$= \int_0^{\sqrt{Y}} dx = \sqrt{Y}$

$F(Y) = \begin{cases} \sqrt{Y} & 0 < Y < 1 \\ 0 & \text{ELSEWHERE} \\ 1 & Y \geq 1 \end{cases}$

$f(Y) = \begin{cases} \frac{1}{2} Y^{-\frac{1}{2}} & ; 0 < Y < 1 \\ 0 & ; \text{ELSEWHERE} \end{cases}$

(1-49) $f(x) = \begin{cases} \frac{x}{6} & ; x = 1, 2, 3 \\ 0 & ; \text{ELSEWHERE} \end{cases}$

$\underline{Y} = \underline{X}^2$

BY INSPECTION:

$f(Y) = \begin{cases} \frac{\sqrt{Y}}{6} & ; Y = 1, 4, 9 \\ 0 & ; \text{ELSEWHERE} \end{cases}$

(1-50) $h > 0$

$Pr[b-h < X \leq b] = F(b) - F(b-h) \geq 0$

$F(b) - F(b-h) \geq F(b)$

$\therefore 0 \leq Pr[b-h \leq X \leq b] \leq F(b)$

AND

$0 \leq \lim_{h \to 0} Pr[b-h < x \leq b] \leq F(b)$

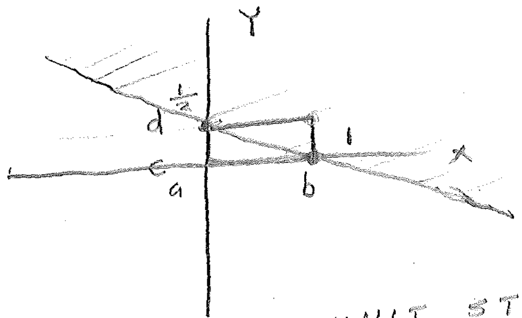
$1 \geq F(b-h) \geq 0 \quad \forall\ h \geq 0$

SINCE $F(x)$ IS NON-DECREASING

$F(b-h) \leq F(b)$

(1-51) $X + 2Y \geq 1$

CONSIDER $X + 2Y = 1 \Rightarrow Y = \frac{1}{2} - \frac{1}{2}X$



UNIT STEP

$F(X,Y) = \mu[X + 2Y]$

FOR THE POINTS SHOWN

$F(b,d) - F(a,d) - F(b,c) + F(a,c) -$

$= -1 < 0 \Rightarrow$ NO GOOD

(1-52)

$F(m) = \frac{1}{2}$

M IS MEDIAN IF

① $F[X < m] \le \frac{1}{2}$ $F[X \le m] \ge \frac{1}{2}$ ②

② IS OBVIOUSLY MET. IF

WE REWRITE ① AS

$F[X \le \lim_{\epsilon \to 0} m + \epsilon]$

AND THE "SHOW THAT"

IS COMPLETED

(1-53) $f(x) = \begin{cases} \frac{1}{3} & ; \ -1 < x < 2 \\ 0 & ; \ \text{ELSEWHERE} \end{cases}$

$$f(|x|) = \begin{cases} \frac{2}{3} & 0 \leq x < 1 \\ \frac{1}{3} & 1 < x < 2 \\ 0 & \text{ELSEWHERE} \end{cases}$$

$$Y = X^2 = |X|^2$$

$$F_Y(Y) = P[Y \leq Y] = P[|X|^2 \leq Y]$$

$$= P[|X| \leq \sqrt{Y}]$$

$$= \int_{-\infty}^{\sqrt{Y}} f(|x|)\, dx$$

$$= \begin{cases} 0 & ; \ Y < 0 \\ \int_0^{\sqrt{Y}} \frac{2}{3}\, dY & ; \ 0 < \sqrt{Y} < 1 \\ \int_1^{\sqrt{Y}} \frac{1}{3}\, dY + Y_0 & ; \ 1 < \sqrt{Y} < 2 \\ 1 & \ Y > 2 \end{cases}$$

$$= \begin{cases} 0 & ; \ Y < 0 \\ \frac{2}{3}\sqrt{Y} & \ 0 < Y < 1 \\ \frac{2}{3} + \frac{1}{3}[\sqrt{Y} - 1] & ; \ 1 < Y < 4 \\ 1 & \ Y > 4 \end{cases}$$

$$= \begin{cases} 0 & Y < 0 \\ \frac{2}{3}\sqrt{Y} & 0 < Y < 1 \\ \frac{1}{3} + \frac{1}{3}\sqrt{Y} & 1 < Y < 4 \\ 1 & Y > 4 \end{cases}$$

$$f(Y) = \frac{d}{dY} F(Y) = \begin{cases} \frac{1}{3\sqrt{Y}} & ; \ 0 < Y < 1 \\ \frac{1}{6\sqrt{Y}} & ; \ 1 < Y < 4 \\ 0 & ; \ \text{ELSEWHERE} \end{cases}$$

(1-54)

$$\left.\begin{array}{l}6 \text{ RED}\\ 7 \text{ WHITE}\\ 3 \text{ BLUE}\end{array}\right\} 16 \text{ CHIPS}$$

W/O REPLACEMENT

(a) P[EACH OF 4 CHIPS IS RED]

$$= \frac{6}{16}\cdot\frac{5}{15}\cdot\frac{4}{14}\cdot\frac{3}{13} = \frac{1\cdot 3}{28\cdot 13} = \frac{3}{364}$$

$$= \frac{\binom{6}{4}}{\binom{16}{4}}$$

(b) P[NONE IS RED] $= \dfrac{\binom{10}{4}}{\binom{16}{4}}$

(c) P[1 CHIP OF EACH COLOR]

= P[2 RED, 1 WHITE, 1 BLUE]

+ P[1 RED, 2 WHITE, 1 BLUE]

+ P[1 RED, 1 WHITE, 2 BLUE]

$$= \frac{\binom{6}{2}\binom{7}{1}\binom{3}{1} + \binom{6}{1}\binom{7}{2}\binom{3}{1} + \binom{6}{1}\binom{7}{1}\binom{3}{2}}{\binom{16}{4}}$$

(1-55)

P[WIN AT LEAST ONE PRIZE]

= 1 - P[WIN NO PRIZES]

$$= 1 - \frac{\binom{990}{5}\binom{10}{0}}{\binom{1000}{5}}$$

(1-56)(a)

P[6 SPADES, 4 HEARTS, 2 DIAM, 1 CL]

$$= \frac{\binom{13}{6}\binom{13}{4}\binom{13}{2}\binom{13}{1}}{\binom{52}{13}}$$

(b) P[ALL CARDS FROM SAME SUIT]

$$= \frac{\binom{4}{1}\binom{13}{13}\binom{39}{0}}{\binom{52}{13}}$$

(1-57) $I = 1, 2, 3, \ldots, 20$

(a) $P[\text{ODD}] = 1/2$

$P[\text{EVEN}] = 1/2$

10 EVEN

10 ODD

$P[\Sigma \text{ IS EVEN}]$

$= P[2 \text{ ARE ODD } \& \text{ } 1 \text{ IS EVEN}]$

$+ P[\text{ALL ARE EVEN}]$

$$= \frac{\binom{10}{2}\binom{10}{1} + \binom{10}{3}\binom{10}{0}}{\binom{20}{3}}$$

(b) $P[\text{PRODUCT IS EVEN}]$

$= 1 - P[\text{PRODUCT IS ODD}]$

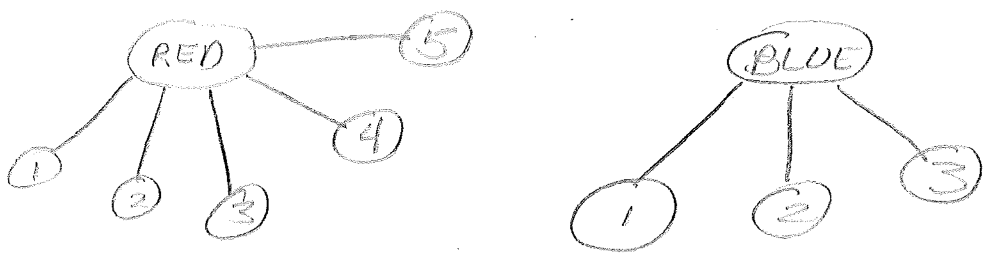
$= P[\text{AT LEAST 1 IS EVEN}]$

$= 1 - P[\text{NONE ARE EVEN}]$

$= 1 - P[\text{ALL ARE ODD}]$

$$= 1 - \frac{\binom{10}{3}\binom{10}{0}}{\binom{20}{3}}$$

(1-58)



DRAW 2 CHIPS

P[SAME NUMBER OR SAME COLOR]
= P[SAME NUMBER]
+ P[SAME COLOR]

$$= \frac{\binom{3}{1}\binom{5}{1}\binom{3}{1}}{\binom{8}{2}} + \frac{\binom{3}{0}\binom{5}{2} + \binom{3}{2}\binom{5}{0}}{\binom{8}{2}}$$

$$= \frac{\binom{3}{1}\binom{5}{1}\binom{3}{1} + \binom{3}{0}\binom{5}{2} + \binom{3}{2}\binom{5}{0}}{\binom{8}{2}}$$

(1-59) $f(x) = \frac{1}{5}$; $x = -2, -1, 0, 1, 2$

$Y = X^2$

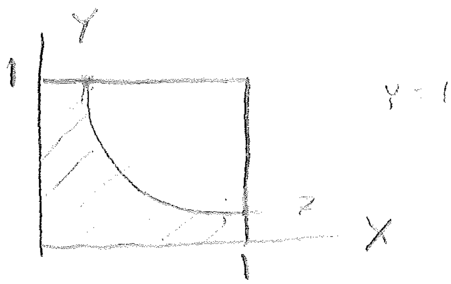
BY INSPECTION

$$f(Y) = \begin{cases} \frac{2}{5} & Y = 1, 4 \\ \frac{1}{5} & Y = 0 \\ 0 & \text{OTHERWISE} \end{cases}$$

(1-60) $f(x,y) = \begin{cases} 1 & ; \ 0 < x, y < 1 \\ 0 & ; \ \text{ELSEWHERE} \end{cases}$

$Z = XY$

$$F(z) = P[Z \le z]$$
$$= P[XY \le z]$$



$$F(z) = 1 - P[XY > z]$$
$$= 1 - \int_z^1 \int_{\frac{z}{y}}^1 dx\, dy$$
$$= 1 - \int_z^1 \left(1 - \frac{z}{y}\right) dy$$
$$= 1 - (y - z \ln y)\Big|_z^1$$
$$= 1 + (z \ln y - y)\Big|_z^1$$
$$= 1 + [(-1) - (z \ln z - z)]$$
$$= z - z \ln z \qquad 0 \le z \le 1$$

$$f(z) = 1 - \ln z - 1$$
$$= -\ln z$$

(1-61) DRAW 13 CARDS

X = # OF SPADES

FIND pdf OF X

$$p[X=n] = \frac{\binom{13}{n}\binom{39}{13-n}}{\binom{52}{13}}$$

Y = # OF HEARTS

$$p[X=n, Y=m]$$

$$= \frac{\binom{13}{n}\binom{13}{m}\binom{26}{13-n-m}}{\binom{52}{13}}$$

$$p[X=2, Y=5]$$

$$= \binom{13}{2}\binom{13}{5}\binom{26}{6} \Big/ \binom{52}{13}$$

(1-62) INT = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

ORDER: $X_1 < X_2 < X_3 < X_4$

THEN $X = X_2$

FIND $p(x)$

$$p(x=1) = p(x=9) = p(x=10) = 0$$

$$p(x=2) = p(X_1 = 1 \ \underline{\text{AND}} \ X_2 = 2)$$

$$= \frac{\binom{2}{2}\binom{8}{2}}{\binom{10}{4}}$$

$$P(X=3) = P(X_1 = 1 \text{ OR } 2 \ \underline{\text{AND}} \ X_2 = 3)$$

$$= \left[\binom{2}{2} + \binom{2}{2}\right]\binom{8}{2} \Big/ \binom{10}{4}$$

$$P(X=4) = P(X_1 = 1 \text{ OR } 2 \text{ OR } 3 \ \underline{\text{AND}} \ X_2 = 4)$$

IN GENERAL

$$P(X = n) = P[X_1 < n \ \underline{\text{AND}} \ X_2 = n]$$

$$= n\binom{2}{2}\binom{8}{2} \Big/ \binom{10}{4}$$

$n = 2, 3, 4, 5, 6, 7, 8$

(1-63)

50 BULBS

48 GOOD 2 BAD

PICK OUT 5 BULBS

(a) P[AT LEAST 1 BAD BULB]

= 1 - P[NO BAD BULBS]

$$= 1 - \frac{\binom{48}{5}\binom{2}{0}}{\binom{50}{5}}$$

(b) FIND $n \ni$ P[FINDING AT LEAST ONE BAD BULB] $\geq \frac{1}{2}$

= 1 - P[FINDING NO BAD BULBS] $< \frac{1}{2}$

OR

P[FINDING NO BAD BULBS]

$$\Rightarrow \frac{\binom{48}{n}\binom{2}{0}}{\binom{50}{n}} < \frac{1}{2}$$

$$\frac{48!\,\cancel{n!}\,(50-n)!}{\cancel{n!}\,(48-n)!\,50!} = \frac{(50-n)(49-n)}{50\cdot 49} < \frac{1}{2}$$

$$(50-n)(49-n) < \frac{50\cdot 49}{2}$$

$$(50-n)(49-n) < 1225$$

$$2450 - 99n + n^2 < 1225$$

$$n^2 - 99n < -1225$$

$$99n - n^2 > 1225$$

$$n(99-n) > 1225$$

n	$n(99-n)$
15	1260
14	1190

$\Rightarrow$ MUST CHOOSE 15 BULBS

(1) $B_1 \subset B_2 \subset B_3 \subset \ldots$

$$B = \bigcup_{n=1}^{\infty} B_n = \lim_{K\to\infty} B_K$$

$$\ni B_K = \bigcup_{i=1}^{K} B_i$$

SHOW $P(B) = \lim_{K\to\infty} P(B_K)$

LET $A_1 = B_1$

$A_2 = B_2 - B_1 = B_2 \cap B_1^*$

$\vdots$

$A_n = B_n - B_{n-1}$

$$B_K = \bigcup_{i=1}^{K} B_i = \bigcup_{i=1}^{K} A_i$$

$A_i \cap A_j = \phi \ \forall i \neq j$

$$\Rightarrow P[B_K] = \sum_{i=1}^{K} P(A_i)$$

$$\lim_{K\to\infty} P[B_K] = P\left[\sum_{i=1}^{\infty} A_i\right]$$

$$= P[B_1 + (B_2 - B_1) + \ldots (B_K - B_{K-1})]$$

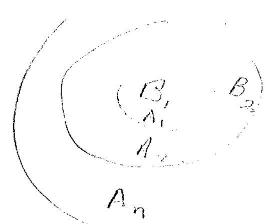
$$= \lim_{K\to\infty} P[B_K]$$

(2) $B_1 \supset B_2 \supset B_3 \ldots$

$$B = \bigcap_{n=1}^{\infty} B_n = \lim_{K\to\infty} B_K$$

SHOW

$P(B) = \lim_{K\to\infty} P(B_K)$



$A_1 = B_1$

$A_2 = B_2 + B_1$

$\vdots$

$A_n = B_n + B_{n-1}$

$B_K = A_1 \cup A_2 \cup A_3 \ldots$

$P[B_K] = P(A_1 \cup A_2 \cup A_3 \ldots A_K] = P[A_K]$

$= P[A_1 + A_2 + \ldots]$

$\lim_{K\to\infty} P[B_K] = P[B]$

9/23/76

(3-1) $M(t) = \left(\frac{1}{3} + \frac{2}{3}e^{t}\right)^{5}$

FOR BINOMIAL

$M(t) = [(1-p) + pe^{t}]^{n}$

$\Rightarrow p = 2/3, \quad n = 2$

$\Rightarrow f(y) = \binom{n}{y} p^{y}(1-p)^{n-y}$

$= \binom{5}{y}\left(\frac{2}{3}\right)^{y}\left(\frac{1}{3}\right)^{5-y}$

$P[Y = 2 \text{ OR } 3] = \binom{5}{2}\left(\frac{2}{3}\right)^{2}\left(\frac{1}{3}\right)^{3} + \binom{5}{3}\left(\frac{2}{3}\right)^{3}\left(\frac{1}{3}\right)^{2}$

$= \frac{5\cdot\cancel{4}^{2}\cdot\cancel{3}}{\cancel{2}\cdot\cancel{3}}\,\frac{4\cdot\cancel{2}}{\cancel{2}\cdot 81} + \frac{5\cdot\cancel{4}^{2}}{3\cdot\cancel{2}}\,\frac{8}{3\cdot 81}$

$= \frac{40}{81} + \frac{80}{9\cdot 81}$

$81 \times 3 = 243$

$= \frac{360+80}{9\cdot 81} = \frac{420}{9\cdot 81} = \frac{140}{3\cdot 81} = \frac{140}{243}$

(3-2) $\sigma^2 = npq = 9\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = 2$

$\mu = np = 3$

$Pr[\mu - 2\sigma < X < \mu + 2\sigma]$

$= Pr[$

BOB MARKS

(1-64) $f(x) = \frac{1}{18}(x+2) \quad ; -2 < x < 4$

$$E[X] = \frac{1}{18}\int_{-2}^{4}[x^2 + 2x]dx$$
$$= \frac{1}{18}\left[\frac{1}{3}x^3 + x^2\right]_{-2}^{4}$$
$$= \frac{1}{18}\left[\frac{1}{3}(64+8) + (16-4)\right]$$
$$= \frac{1}{18}\left[\frac{72}{3} + 12\right] = \frac{1}{18}[24+12]$$
$$= \frac{36}{18} = 2$$

$$E[(X+2)^3] = \frac{1}{18}\int_{-2}^{4}(x+2)^4 dx$$
$$= \frac{1}{18}\,\frac{1}{5}(x+2)^5\Big|_{-2}^{4}$$
$$= \frac{1}{5\cdot 18}\left[6^5 - 0^5\right]$$
$$= \frac{6^4}{15} = \frac{2\cdot\cancel{3}\cdot 6^3}{\cancel{3}\cdot 5}$$
$$= \frac{2\,6^3}{5} = \frac{432}{5}$$

$\begin{array}{r} 36 \\ 6 \\ \hline 216 \end{array}$

$$E[6X - 2(X+2)^3]$$
$$= 6E(X) - 2E[(X+2)^3]$$
$$= 6(2) - 2\cdot\frac{432}{5}$$
$$= 12 - \frac{864}{5}$$
$$= \frac{60-864}{5} = -\frac{804}{5}$$

(1-65)

$f(x) = \frac{1}{5}$, $x = 1, 2, 3, 4, 5$

$E[X] = \sum_{n=1}^{5} \frac{x}{5}$

$= \frac{1}{5}[1+2+3+4+5]$

$= \frac{15}{5} = 3$

$E[X^2] = \frac{1}{5}[1+4+9+16+25]$

$= \frac{1}{5}(55] = 11$

$E[(X+2)^2] = E[X^2] + 4E(X) + E(4)$

$= 11 + 4(3) + 4$

$= 11 + 12 + 4$

$= 27$

(1-66) $f(x,y) = \frac{1}{3}$ $(x,y) = (0,0),(0,1)$
$(1,1)$

$$E[(X-\tfrac{1}{3})(Y-\tfrac{2}{3})]$$
$$= \frac{1}{3}\left[(0-\tfrac{1}{3})(0-\tfrac{2}{3}) + (0-\tfrac{1}{3})(1-\tfrac{2}{3}) + (1-\tfrac{1}{3})(1-\tfrac{2}{3})\right]$$
$$= \frac{1}{3}\left[\frac{2}{9} - \frac{1}{9} + \frac{2}{9}\right]$$
$$= \frac{1}{3}\left[\frac{3}{9}\right] = \frac{1}{9}$$

(1-67) $f(x,y) = e^{-x-y}$; $0 < x < \infty$; $0 < y < \infty$

$\mu(X,Y) = X$

$v(X,Y) = Y$

$w(X,Y) = XY$

$E[\mu(X,Y)] = E[X] = \int_0^\infty e^{-y} dy \int_0^\infty x e^{-x} dx$

$u = x \qquad dv = e^{-x} dx$

$du = dx \qquad v = -e^{-x}$

$E[x] = -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x} dx$

$= 0 + 1 = \underline{1}$

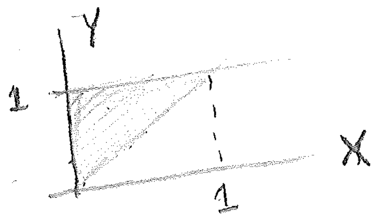
$E[v(x,y)] = E[Y] = \underline{1}$

$E[w(x,y)] = E[xy]$

$= \int_0^\infty x e^{-x} dx \int_0^\infty y e^{-y} dy$

$= (1)(1) = 1$

(1-68) $f(x) = 2 \; ; \quad \begin{matrix} 0 < x < Y \\ 0 < Y < 1 \end{matrix}$



$$E[U(X,Y)] = E[X]$$
$$= \int_0^1 \int_{x=0}^{Y} x\,2\,dx\,dY$$
$$= \int_0^1 x^2 \Big|_0^Y dY = \int_0^1 Y^2 dY = \frac{1}{3}$$

$$E[V(X,Y)] = E[Y]$$
$$= 2\int_0^1 Y \left[\int_0^Y dx\right] dY$$
$$= 2\int_0^1 Y^2 dY = 2/3$$

$$E[W(X,Y)] = E[XY]$$
$$= 2\int_0^1 Y \int_0^Y X\,dx\,dY$$
$$= \int_0^1 Y^3 dY = \frac{1}{4} Y^4 \Big|_0^1 = \frac{1}{4}$$
$$\neq \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$$

(1-69) $f(x) > 0 \quad ; -1, 0, 1$

$\qquad\qquad = 0 \quad ;$ OTHERWISE

$f(0) = \frac{1}{2}$

$\Rightarrow f(-1) + f(1) = \frac{1}{2}$

$E[X^2] = 0\frac{1}{2} + f(-1)(-1)^2 + f(1)(1)^2$

$= f(-1) + f(1) = \frac{1}{2}$

$f(0) = \frac{1}{2}$

$E[X] = \frac{1}{6} = f(1) - f(-1)$

THUS

$f(1) - f(-1) = \frac{1}{6}$

$\underline{f(1) + f(-1) = \frac{1}{2} = 3/6}$

$2f(1) = 4/6$

$f(1) = 4/12 = 1/3$

$f(-1) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

(1-70) $\begin{matrix} 8 \\ \#2 \end{matrix}$ $\begin{matrix} 2 \\ \#5 \end{matrix}$

CHOOSE 3 (W/O REPLACEMENT)

LET X = # OF #5 CHIPS

$\Rightarrow u(x) = 5x + 2(3-x) = 6 + 3x \; ; x = 0,1,2$

$$P(x) = \frac{\binom{2}{x}\binom{8}{3-x}}{\binom{10}{3}} \quad ; \; x = 0,1,2$$

$$E[u(x)] = \frac{1}{\binom{10}{3}}\Big[(6+0)\binom{2}{0}\binom{8}{3} + (6+3)\binom{2}{1}\binom{8}{2} + (6+6)\binom{2}{2}\binom{8}{1}\Big]$$

$$= \frac{1}{\binom{10}{3}}\Big[\frac{\cancel{6}\cdot 8\cdot 7\cdot 6}{\cancel{3\cdot 2}} + \frac{9\cdot\cancel{2}\cdot 8\cdot 7}{\cancel{2\cdot 1}} + 12\cdot 1\cdot 8\Big]$$

$$= \frac{1}{\binom{10}{3}}\big[336 + 504 + 96\big]$$

$$= \frac{1}{\binom{10}{3}}\big[936\big]$$

$$\binom{10}{3} = \frac{10\cdot\cancel{9}^{3}\cdot\cancel{8}^{4}}{\cancel{3\cdot 2}} = 120$$

$$= \frac{936}{120} = \$7.80$$

(1-71)

$$E[|X-b|] = \int |x-b|\, f(x)\,dx$$

$$= \int_{-\infty}^{b} (b-x) f(x)\,dx + \int_{b}^{\infty} (x-b) f(x)\,dx$$

$$= \int_{-\infty}^{b} b f(x)\,dx - \int_{b}^{\infty} b f(x)\,dx$$

$$+ \int_{b}^{\infty} x f(x)\,dx - \int_{-\infty}^{b} x f(x)\,dx$$

$$E[|X-m|] = \int_{-\infty}^{m} m f(x)\,dx - \int_{m}^{\infty} m f(x)\,dx$$

$$+ \int_{m}^{\infty} x f(x)\,dx - \int_{-\infty}^{m} x f(x)\,dx$$

$$E[|X-b|] - E[|X-m|]$$

$$= \int_{-\infty}^{b} b f(x)\,dx + \int_{m}^{-\infty} m\, \cancel{f(x)}\,dx \quad \tfrac{m}{2}$$

$$- \left[\int_{b}^{\infty} b f(x)\,dx + \int_{\infty}^{m} m f(x)\,dx\right]$$

$$+ \left[\int_{b}^{\infty} x f(x)\,dx + \int_{\infty}^{m} x f(x)\,dx\right]$$

$$- \left[\int_{-\infty}^{b} x f(x)\,dx + \int_{m}^{-\infty} x f(x)\,dx\right]$$

$$= \int_{-\infty}^{b} b f(x)\,dx - \int_{b}^{\infty} b f(x)\,dx$$

$$+ \int_{b}^{m} x f(x)\,dx - \int_{m}^{b} x f(x)\,dx$$

$$= \int_{-\infty}^{b} b f(x)\,dx + \int_{\infty}^{b} b f(x)\,dx$$

$$+ 2\int_{m}^{b} x f(x) \quad \int_{m}^{-\infty} b f(x)\,dx\Big]$$ ← SUBTRACTING b/2

$$= \left[\int_{-\infty}^{b} b f(x)\,dx + \int_{m} \quad \int_{m}^{\infty} b f(x)\,dx\right]$$ ← ADDING b/2

$$+ \left[\int_{\infty}^{b} b f(x)\,dx + \int_{m}\right.$$

$$- 2\int_{m}^{b} x f(x)\,dx \Big]$$

$$= \left[\int_{m}^{b} b f(x)\,dx + \int_{m}^{b} b f(x)\,dx\right]$$

$$- 2\int_{m}^{b} x f(x)\,dx$$

$$= 2\int_{m}^{b} b f(x)\,dx - 2\int_{m}^{b} x f(x)\,dx$$

$$= 2\int_{m}^{b} (b-x) f(x)$$

$$\Rightarrow E[|X-b|] = E[|X-m|] + 2\int_{m}^{b} (b-x) f(x)$$

EXTRA

$E[(x-b)^2] = \int_{-\infty}^{\infty} (x-b)^2 f(x)dx$

$= \int x^2 f(x)dx - 2b\int x f(x)dx + b^2$

$E[(x-m)^2] = \int x^2 f(x)dx - 2m\int x f(x)dx + m^2$

$E[(x-b)^2] - E[(x-m)^2]$

$= -2b\int x f(x)dx + b^2$

$+ 2m\int x f(x)dx - m^2$

$= 2(m-b)\int x f(x)dx + (b^2 - m^2)$

$= 2(m-b)\mu + (b^2 - m^2)$

$\Rightarrow E[(x-b)^2] - E[(x-m)^2]$

$= 2(m-b)\mu + (b^2 - m^2)$

(1-72) (a) $f(x) = 2x, \quad 0 < x < 1$

$$E[\sqrt{X}] = 2\int_0^1 x^{3/2}\,dx$$
$$= 2\,\frac{2}{5}\,x^{5/2}\Big|_0^1 = \frac{4}{5}$$

(b) $Y = \sqrt{X} > 0$

$$F_Y(y) = P[\sqrt{X} \le y]$$
$$= P[X \le y^2] = \int_0^{y^2} 2x\,dx$$
$$= x^2\Big|_0^{y^2} = y^4; \quad 0 < y < 1$$
$$1; \quad y > 1$$

$$f(y) = 4y^3$$

(c) $$E[Y] = 4\int_0^1 y^4\,dy$$
$$= \frac{4}{5}\,y^5\Big|_0^1 = 4/5$$

(SAME AS a)

(1-73) 1 2 3 4 5 6

FIND $f(x)$ OF $Y = |X_1 - X_2|$

$Y = 1 \Rightarrow \begin{cases} (2,1)\ (3,2)\ (4,3)\ (5,4)\ (6,5) \\ (1,2)\ (2,3)\ (3,4)\ (4,5)\ (5,6) \end{cases}$

$Y = 2 \Rightarrow \begin{cases} (3,1)\ (4,2)\ (5,3)\ (6,4) \\ (1,3)\ (2,4)\ (3,5)\ (4,6) \end{cases}$

$Y = 3 \Rightarrow \begin{cases} (4,1)\ (5,2)\ (6,3) \\ (1,4)\ (2,5)\ (3,6) \end{cases}$

$Y = 4 \Rightarrow \begin{cases} (5,1)\ (6,2) \\ (1,5)\ (5,1) \end{cases}$

$Y = 5 \Rightarrow \begin{cases} (6,1) \\ (1,6) \end{cases}$

$N = 10 + 8 + 6 + 4 + 2 = 30$

$P(Y=1) = \frac{10}{30}$ $\quad P(Y=4) = \frac{4}{30}$

$P(Y=2) = \frac{8}{30}$ $\quad P(Y=5) = \frac{2}{30}$

$P(Y=3) = 6/30$

$$E(Y) = (1)\left(\frac{10}{30}\right) + (2)\left(\frac{8}{30}\right) + (3)\left(\frac{6}{30}\right) + (4)\left(\frac{4}{30}\right) + 5\left(\frac{2}{30}\right)$$
$$= \frac{1}{30}(10 + 16 + 18 + 16 + 10)$$
$$= \frac{1}{30}(20 + 32 + 18) = \frac{70}{30} = 7/3$$
$$= 2\tfrac{1}{3}$$

(1-74)(a) $f(x) = \frac{3!}{x!(3-x)!}\left(\frac{1}{8}\right) \quad x = 0,1,2,3$

$f(1) = \frac{6}{1 \cdot 2}\frac{1}{8} = 3/8$

$f(0) = \frac{1}{8}$

$f(2) = \frac{6}{2 \cdot 1}\frac{1}{8} = \frac{3}{8}$

$f(3) = \frac{1}{8}$

$E(x) = \frac{1}{8}(0) + \frac{3}{8}(1) + \frac{3}{8}(2) + \frac{1}{8}(3)$

$= \frac{1}{8}[3 + 6 + 3] = \frac{12}{8} = 3/2$

$E[(x)^2] = \frac{1}{8}0^2 + \frac{3}{8}(1)^2 + \frac{3}{8}(4) + \frac{1}{8}9$

$= \frac{1}{8}[3 + 12 + 9] = \frac{24}{8} = 3$

$\sigma^2 = 3 - \frac{9}{4} = \frac{3}{4}$

(b) $f(x) = 6x(1-x) \quad ; 0 < x < 1$

$= 6x - 6x^2$

$E[x] = \int_0^1 (6x^2 - 6x^3)dx$

$= 2x^3 - \frac{6}{4}x^4 \Big|_0^1 = 2 - \frac{6}{4} = 2 - \frac{3}{2} = \frac{1}{2}$

$E[x^2] = \int_0^1 [6x^3 - 6x^4]dx$

$= \frac{6}{4}x^4 - \frac{6}{5}x^5 \Big|_0^1$

$= \frac{6}{4} - \frac{6}{5} = 6\left(\frac{1}{4} - \frac{1}{5}\right) = 6\frac{1}{20} = \frac{3}{10}$

$\sigma^2 = \frac{3}{10} - \frac{1}{4} = \frac{12-10}{40} = \frac{2}{40} = \frac{1}{20}$

(c) $f(x) = \frac{2}{x^3} \quad ; 1 < x < \infty$

$E[x] = 2\int_1^\infty x^{-2}dx$

$= -2x^{-1}\Big|_1^\infty = 0 - (-2) = 2$

$E[x^2] = 2\int_1^\infty x^{-1}dx$

$= 2\ln x\Big|_1^\infty$

$\Rightarrow E[x^2]$ IS UNDEFINED

(1-75) $f(x) = \left(\frac{1}{2}\right)^x \quad ; x = 1, 2, 3, \ldots$

$$M(t) = E[e^{tx}]$$
$$= \sum_{n=1}^{\infty} e^{tx}\left(\frac{1}{2}\right)^x$$
$$= \sum_{n=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$
$$= \sum_{n=0}^{\infty} \left(\frac{e^t}{2}\right)^x - 1$$

USE GEOMETRIC SERIES

$$M(t) = \frac{1}{1 - \frac{e^t}{2}} - 1 \qquad ; \frac{e^t}{2} < 1$$
$$= \frac{e^t/2}{1 - e^t/2} \qquad t < \ln 2$$
$$= e^t/2 - e^t \qquad t < \ln 2$$

$$= \frac{1}{2e^{-t} - 1} = (2e^{-t} - 1)^{-1} \quad ; t < \ln 2$$

$$\mu = \frac{d}{dt}M(0) = (-1)(2e^{-t} - 1)^{-2}(-2e^{-t})\Big|_0$$
$$= \frac{2e^{-t}}{(2e^{-t} - 1)^2}\Big|_0$$
$$= \frac{2}{(2-1)^2} = 2$$

$$E(X^2) = \frac{d^2}{dt^2}M(0) = \frac{(2e^{-t}-1)^2(-2e^{-t}) - 2(2e^{-t}-1)(-2e^{-t})2e^{-t}}{[2e^{-t}-1]^4}\Bigg|_0$$

$$= \frac{(2-1)^2(-2) - 2(1)(-2)(2)}{1^4}$$
$$= -2 + 8 = 6$$

$$\sigma^2 = E(X^2) - \mu^2 = 6 - 4 = 2$$

(1-76) (a) $f(x) = 6x(1-x) = c(x - x^2)$

$$\mu = E[x] = 6\int_0^1 (x^2 - x^3)\,dx$$
$$= 6\left[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_0^1$$
$$= 6\left[\tfrac{1}{3} - \tfrac{1}{4}\right] = \tfrac{6}{12} = \tfrac{1}{2}$$

$$E(x^2) = 6\int_0^1 (x^3 - x^4)\,dx$$
$$= 6\left[\tfrac{1}{4}x^4 - \tfrac{1}{5}x^5\right]_0^1 = \tfrac{6}{20} = \tfrac{3}{10}$$

$$\sigma^2 = E(x^2) - E(x)^2$$
$$= \tfrac{3}{10} - \tfrac{1}{4} = \tfrac{12-10}{40} = \tfrac{1}{20} \Rightarrow \sigma = \tfrac{1}{\sqrt{20}}$$

$$P[\mu - 2\sigma < x < \mu + 2\sigma]$$

$$= P\left[\tfrac{1}{2} - \tfrac{2}{\sqrt{20}} < x < \tfrac{1}{2} + \tfrac{2}{\sqrt{20}}\right]$$
$$= P[-0.052 < x < 0.94]$$

$$= 6\int_{\frac{1}{2}-\frac{2}{\sqrt{20}}}^{\frac{1}{2}+\frac{2}{\sqrt{20}}} (x - x^2)$$

$$= 6\left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_{\frac{1}{2}-\frac{2}{\sqrt{20}}}^{\frac{1}{2}+\frac{2}{\sqrt{20}}}$$

$$= 3x^2 - 2x^3\Big|_{\frac{1}{2}-\frac{2}{\sqrt{20}}}^{\frac{1}{2}+\frac{2}{\sqrt{2}}}$$

$$= (2.6916 - 0.8499) - (0.0084 - 0.0003)$$
$$= 1.8336$$

(b) $f(x) = \left(\frac{1}{2}\right)^x \quad x = 1, 2, 3, \ldots$

$$E[X] = \sum_{n=1}^{\infty} x \left(\frac{1}{2}\right)^x$$

FROM (1-75):

$\sigma^2 = 2$, $\mu = 2$

$$Pr[\mu - 2\sigma < X < \mu + 2\sigma]$$

$$= P[\underbrace{2 - 2\sqrt{2}}_{<0} < X < 2 + 2\sqrt{2}]$$

$$= P[1 \leq X < 4.8]$$

$$= P[1 \leq X \leq 4]$$

$$= \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4$$

$$= \frac{1}{16}[8 + 4 + 2 + 1] = 15/16$$

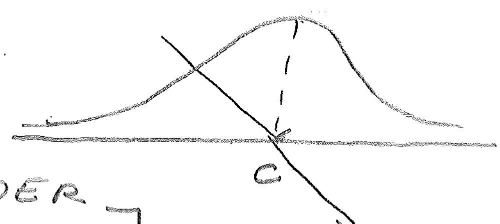
(1-77)
$$\sigma^2 = E[(x-\mu)^2]$$
$$= \int_{-\infty}^{\infty} f(x)(x-\mu)^2 dx > 0$$

BUT
$$\sigma^2 = E[x^2] - E[x]^2 \geq 0$$
$$\Rightarrow E[x^2] \geq E[x]^2$$

(1-78)

$$E[x] = \int_{-\infty}^{\infty} x f(x)dx$$

$$= \int_{-\infty}^{c} x f(x)dx + \int_{c}^{\infty} x f(x)dx$$



CONSIDER

$$E[(x-c)] = E(x) - c$$

$$= \int_{-\infty}^{\infty} (x-c) f(x)dx$$

$$= \int_{-\infty}^{c} (x-c) f(x)dx + \int_{c}^{-\infty} (x-c) f(x)dx$$

IF $f(x)$ IS SYSMMETRIC ABOUT c, $f(x+c) = f(-x+c)$

(i.e, $f(x)$ IS "EVEN" ABOUT c)

also $x-c$ IS "ODD" ABOUT c

$\Rightarrow (x-c) f(x)$ IS "ODD" ABOUT c

$$\Rightarrow \int_{-\infty}^{c} (x-c) f(x)dx = -\int_{c}^{\infty} (x-c) f(x)dx$$

AND

$$E[(x-c)] = 0 = E[x] - c$$

$$\Rightarrow c = E[x]$$

(1-79) $$E\left[\frac{x-\mu}{\sigma}\right]=\int_{-\infty}^{\infty}\frac{x-\mu}{\sigma}f(x)dx$$
$$=\frac{1}{\sigma}\int_{-\infty}^{\infty}[x-\mu]f(x)dx$$
$$=\frac{1}{\sigma}\left[\int x f(x)dx-\mu\right]$$
$$=\frac{1}{\sigma}[\mu-\mu]=0$$

$$E\left[\left(\frac{x-\mu}{\sigma}\right)^2\right]=\int\left(\frac{x-\mu}{\sigma}\right)^2 f(x)dx$$
$$=\frac{1}{\sigma^2}\left[\int x^2 f(x)dx-2\mu\int x f(x)dx+\mu^2\right]$$
$$=\frac{1}{\sigma^2}\left[E(x^2)-2\mu^2+\mu^2\right]$$
$$=\frac{1}{\sigma^2}\left[E(x^2)-\mu^2\right]=\frac{1}{\sigma^2}\sigma^2=1$$

$$E\left[e^{t\left(\frac{x-\mu}{\sigma}\right)}\right]=\int_{-\infty}^{\infty}e^{t\left(\frac{x-\mu}{\sigma}\right)}f(x)dx$$
$$=e^{-\frac{\mu}{\sigma}t}\int e^{\frac{tx}{\sigma}}f(x)dx$$
$$=e^{-\frac{\mu}{\sigma}t}\int e^{\left(\frac{t}{\sigma}\right)x}f(x)dx$$
$$=e^{-\frac{\mu}{\sigma}t}M\left(\frac{t}{\sigma}\right)$$
$$\ni M(t)=\int e^{tx}f(x)dx$$

(1-80) $f(x) = \frac{1}{3}$; $-1 < x < 2$

$M(t) = \frac{1}{3}\int_{-1}^{2} e^{tx}\,dx$

$= \frac{1}{3t} e^{tx}\Big|_{-1}^{2}$ $t \neq 0$

$= \dfrac{e^{2x} - e^{-x}}{3t}$

FOR $t = 0$

$M(t) = \frac{1}{3}\int_{-1}^{2} e^{0} d$

$= \frac{1}{3} x \Big|_{-1}^{2}$

$= \frac{1}{3}[2 - (-1)] = \frac{3}{3} = 1$

(1-81) $E[(x-b)^2] = \int_{-\infty}^{\infty} (x-b)^2 f(x)dx$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx + b^2 - 2b\int_{-\infty}^{\infty} x f(x)dx$$

$$= E(x^2) + b^2 - 2b\int x f(x)dx$$

NOW

$E[(x-b)^2] > 0$

AND $E(x^2)$ IS FIXED.

THUS, TO MINIMIZE $E[(x-b)^2]$, WE MUST MINIMIZE

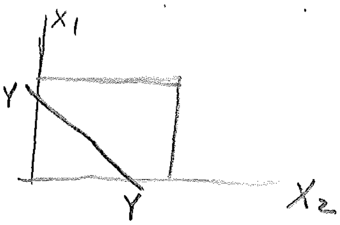
$b^2 - 2b\int x f(x)dx$ WRT b

$$\frac{d}{db}\left[b^2 - 2b\int x f(x)dx\right] = 2b - 2\mu = 0 \Rightarrow b = \mu = E(x)$$

(1-82) $f(x_1, x_2) = 2x_1 \qquad 0 < x_1 < 1$

$\qquad\qquad\qquad\qquad\qquad 0 < x_2 < 1$

$E[X_1 + X_2]$



$Y = X_1 + X_2 \Rightarrow X_1 = Y - X_2$

$$E[X_1 + X_2] = \int_0^1 \int_0^1 2x_1(x_1 + x_2)\,dx_1\,dx_2$$
$$= \int_0^1 \int_0^1 (2x_1^2 + 2x_1x_2)\,dx_1\,dx_2$$
$$= \int_0^1 \tfrac{2}{3}x_1^3 + x_1^2 x_2 \Big|_0^1 dx_2$$
$$= \int_0^1 \left(\tfrac{2}{3} + x_2\right) dx_2$$
$$= \tfrac{2}{3}x_2 + \tfrac{1}{2}x_2^2 \Big|_0^1$$
$$= \tfrac{2}{3} + \tfrac{1}{2} = \frac{4+3}{6} = 7/6$$

$$\sigma^2_{X_1X_2} = E[\{X_1 + X_2 - E(X_1 + X_2)\}^2]$$
$$= E[(X_1 + X_2)^2] - 2E(X_1 + X_2)^2 + E(X_1 + X_2)^2$$
$$= E[(X_1 + X_2)^2] - E(X_1 + X_2)^2$$
$$E[(X_1 + X_2)^2] = 2\int_0^1 \int_0^1 (x_1 + x_2)^2 x_1\,dx_2\,dx_1$$

$$= 2\int_0^1 \tfrac{1}{3}(x_1 + x_2)^3 x_1 \Big|_0^1 dx_1$$
$$= \frac{2}{3}\int (x+1)^3 x_1$$

$$= \frac{2}{3}\int_0^1 [x^4 + 3x^3 + 3x^2 + x]\,dx$$
$$= \frac{2}{3}\left[\frac{1}{4} + \frac{3}{4} + \frac{3}{3} + \frac{1}{2}\right]$$
$$= \frac{2}{3}\left[\frac{5}{2}\right] = 5/3$$

$$\Rightarrow \sigma^2_{X_1X_2} = \frac{5}{3} - \frac{49}{36} = \frac{60 - 49}{36} = \frac{11}{36}$$

(1-83) $f(x,y,z) = e^{-x-y-z}; 0 < x,y,z < \infty$

$$E[e^{t(x+y+z)}] = \iiint_0^\infty e^{-x-y-z} e^{tx+ty+tz} \, dx\,dy\,dz$$

$$= \iiint_0^\infty e^{x(t-1)} dx \; e^{y(t-1)} dy \; e^{z(t-1)} dz$$

$$= \left[\int_0^\infty e^{-x(1-t)} dx \right]^3$$

$$= \frac{1}{(1-t)^3} \quad ; t < 1$$

(1-84)

(I) $E[(x-b)^2]=0 \implies f(x)=\begin{cases}1 & ; x=b \\ 0 & ; \text{OTHERWISE}\end{cases}$

ASSUME $\ni f(x)=a_x \ni x \in I$

$\ni E[(x-b)^2]=0$

$\Rightarrow \sum_x (x-b)^2 a_x = 0$

$\sum_x x^2 a_x - 2\sum_x a_x x b + b^2 \sum_x a_x = 0$

$\sum_x x^2 a_x - 2\sum_x a_x x b + b^2 = 0$

WE HAVE SHOWN THAT THE LEFT HAND SIDE IS MINIMUM FOR $b=\mu$. THUS

$\sum_x x^2 a_x = [\sum_x x\, a_x]^2 \quad ; \sum a_x = 1$

THIS EXPRESSION CAN ACHEIVE EQUALITY IF (THERE IS ONE TERM $\Rightarrow a_x=1$

(II) $f(x)=\begin{cases}1 & ; x=b \\ 0 & ; \text{OTHER}\end{cases} \implies E[(x-b)^2]=0$

$E[x]= b$

$E[x^2]=b^2$

$\sigma^2 = E[(x-b)^2] = E(x^2)-E(x)^2 = b^2-b^2=0$

(1-85) $K(t) = E[t^x]$

$$= \int t^x f(x)\,dx$$

$$\frac{d}{dt} K(t) = \int x\, t^{x-1} f(x)\,dx$$

$$\frac{d^2}{dt^2} K(t) = \int x(x-1)\, t^{x-2} f(x)\,dx$$

$$\vdots$$

$$\frac{d^n}{dt^n} K(t) = \int x(x-1)\cdots(x-n+1)\, t^{x-n} f(x)\,dx$$

$$\frac{d^n}{dt^n} K(1) = \int x(x-1)\cdots(x-n+1) f(x)\,dx$$

$$= E[x(x-1)\cdots(x-n+1)]$$

(1-86) $E[(x-7)] = 3$

$E[(x-7)^2] = 11$

$E[(x-7)^3] = 15$

$E(x-7) = E(x) - 7 = 3 \Rightarrow E(x) = 10$

$E[(x-10)] = E(x) - 10 = 0 \Longleftarrow$

$E[(x-7)^2] = E(x^2) - 14E[x] + 49 = 11$

$E(x^2) - 140 + 49 = 11$

$E(x^2) = 102$

$E[(x-10)^2] = E(x^2) - 20E(x) + 100$

$= 102 - 200 + 100 = 2 \Longleftarrow$

$E[(x-7)^3] = E(x^3) - 21E(x^2) + 147E(x) - 343 = 15$

$E(x^3) = (21)(102) - 1470 + 347 + 15 = 1030$

$E[(x-10)^3]$

$= E(x^3) - 30E(x^2) + 300E(x) - 1000$

$= 1030 - 3060 + 3000 - 1000$

$= -30$

(1-87) $R(t) = E\left[e^{t(x-b)}\right]$

$$= \int e^{t(x-b)} f(x)\,dx$$

$$\frac{d^n}{dt^n} R(t) = \int (x-b)^n e^{t(x-b)} f(x)\,dx$$

$$\frac{d^n}{dt} R(0) = \int (x-b)^n f(x)\,dx$$

$$= E\left[(x-b)^n\right]$$

(1-88) $M(t) = \int e^{tx} f(x)dx$

$\psi(t) = \ln M(t) = \ln \int e^{tx} f(x)dx$

$\frac{d}{dt}\psi(t) = \frac{d}{dt}\ln \int e^{tx} f(x)dx$

$= \frac{\int x e^{tx} f(x)dx}{\int e^{tx} f(x)dx}$

$\frac{d}{dt}\psi(0) = \frac{\int x f(x)dx}{\int f(x)dx} = \frac{E(x)}{1} = \mu$

$\frac{d^2}{dt^2}\psi(t) = \frac{\int e^{tx} f(x)dx \int x^2 e^{tx} f(x)dx - \int x f(x)dx \int x f(x)dx}{\left[\int f(x) e^{tx} dx\right]^2}$

$\frac{d^2}{dt^2}\psi(0) = \frac{(1)\int x^2 f(x)dx - \left[\int x f(x)dx\right]^2}{[1]^2}$

$= E(x^2) - E(x)^2 = \sigma^2$

(1-89) $F(x) = \begin{cases} 0 & ; x < 0 \\ \frac{x}{8} & ; 0 \le x < 2 \\ \frac{x^2}{16} & ; 2 \le x < 4 \\ 1 & ; x \ge 4 \end{cases}$

$f(x) = \begin{cases} \frac{1}{8} & ; 0 \le x < 2 \\ \frac{x}{8} & ; 2 \le x < 4 \\ 0 & ; \text{OTHERWISE} \end{cases}$

$$E(x) = \frac{1}{8}\int_0^2 x\,dx + \frac{1}{8}\int_2^4 x^2\,dx$$
$$= \frac{1}{8}\cdot\frac{1}{2}x^2\Big|_0^2 + \frac{1}{8}\cdot\frac{1}{3}x^3\Big|_2^4$$
$$= \frac{1}{16}(4) + \frac{1}{24}(64-8)$$
$$= \frac{1}{4} + \frac{56}{24} = \frac{1}{4} + \frac{28}{12} = \frac{31}{12}$$

$$\begin{array}{r} 64 \\ 4 \\ \hline 256 \end{array}$$

$$E(x^2) = \frac{1}{8}\int_0^2 x^2\,dx + \frac{1}{8}\int_2^4 x^3\,dx$$
$$= \frac{1}{8}\left[\frac{1}{3}x^3\Big|_0^2 + \frac{1}{4}x^4\Big|_2^4\right]$$
$$= \frac{1}{8}\left[\frac{1}{3}\cdot 8 + \frac{1}{4}(256-16)\right]$$
$$= \frac{1}{8}\left[\frac{8}{3} + \frac{240}{4}\right]$$
$$= \frac{1}{8}\left[\frac{8}{3} + 60\right]$$
$$= \frac{1}{8}\left[\frac{8}{3} + \frac{180}{3}\right]$$
$$= \frac{188}{24} = \frac{94}{12} = \frac{47}{6}$$

$$\begin{array}{r} 31 \\ 31 \\ \hline 31 \\ 93 \\ \hline 961 \end{array} \qquad \begin{array}{r} 94 \\ 12 \\ \hline 188 \\ 94 \\ \hline 1128 \end{array}$$

$$\sigma^2 = E(x^2) - E(x)^2$$
$$= \frac{94}{12} - \frac{(31)^2}{144}$$
$$= \frac{1}{144}(1128 - 961)$$
$$= \frac{1}{144}(167)$$
$$= 167/144$$

(1-90) $M(t) = \frac{1}{(1-t)^3}$

$$\frac{1}{1-t} = (1-t)^{-1} = \sum_{n=0}^{\infty} t^n$$

$$\frac{d}{dt}(1-t)^{-1} = (-1)(-1)(1-t)^{-2} = \sum_{n=1}^{\infty} n t^{n-1}$$

$$= (1-t)^{-2}$$

$$\frac{d}{dt}(1-t)^{-2} = (-2)(-1)(1-t)^{-3} = \sum_{n=2}^{\infty} n(n-1) t^{n-2}$$

$$\Rightarrow (1-t)^{-3} = \frac{1}{2} \sum_{n=2}^{\infty} n(n-1) t^{n-2}$$

$$= \frac{1}{2}\left[(2)(1) + (3)(2)t + \ldots\right]$$

$$\Rightarrow M(0) = 1$$

$$\frac{d}{dt}(1-t)^{-3} = \frac{1}{2} \sum_{n=3}^{\infty} n(n-1)(n-2) t^{n-3}$$

$$= \frac{1}{2}\left[(3)(2)(1) + (4)(3)(2)t + \ldots\right]$$

$$\frac{d}{dt}(1-t)^{-3}\Big|_{t=0} = 3 = E(x^2)$$

$$\Rightarrow \sigma^2 = E(x^2) - E(x)^2$$

$$= 3 - (1)^2 = 2$$

(1-91) $f(x)$ NONZERO FOR $0 < x < b < \infty$

$$E[X] = \int_0^b x\, f(x)\, dx \leftarrow \text{INTEGRATION BY PARTS}$$

$$u = x \qquad dv = f(x)\,dx$$

$$du = dx \qquad v = F(x)$$

$$\Rightarrow E[X] = x\,F(x)\Big|_0^b - \int_0^b F(x)\,dx$$

$$= b\,F(b) - \int_0^b F(x)\,dx$$

$$= b - \int_0^b F(x)\,dx$$

$$= \int_0^b [1 - F(x)]\,dx$$

(1-92) $$E\left[\left(\frac{X-\mu}{c}\right)^{2k}\right] = \int_{-\infty}^{\infty}\left(\frac{x-\mu}{c}\right)^{2k} f(x)\,dx$$

$$= \int_A \left(\frac{x-\mu}{c}\right)^{2k} f(x)\,dx + \int_{A^*}\left(\frac{x-\mu}{c}\right)^{2k} f(x)\,dx \geq 0$$

$$\Rightarrow E\left[\left(\frac{X-\mu}{c}\right)^{2k}\right] \geq \int_A \left(\frac{x-\mu}{c}\right)^{2k} f(x)\,dx$$

LET $A = \left\{\left(\frac{x-\mu}{c}\right)^{2k};\ |x-\mu| \geq c\right\}$

$$\Rightarrow \int_A \left(\frac{x-\mu}{c}\right)^{2k} f(x)\,dx \geq c\int_A f(x)\,dx = c\,P[|x-\mu| \geq c] > P[|x-\mu| \geq c]$$

$$\therefore E\left[\left(\frac{X-\mu}{c}\right)^{2k}\right] \geq P[|X-\mu| \geq c]$$

(1-93) $P_r[X \le 0] = 0$

$\mu = E(x)$

SHOW $P_r[X \ge 2\mu) \le \frac{1}{2}$

THEM 6 $\Rightarrow P_r[u(X) \ge c] \le E(u(x))/c$

LET $u(X) = X$ AND $c = 2\mu$

$$\Rightarrow P_r[X \ge 2\mu] \le \frac{E[X]}{2\mu} = \frac{\mu}{2\mu} = \frac{1}{2}$$

MORE GENERALLY:

$$P_r[X \ge k\mu] \le \frac{E[X]}{k\mu} = \frac{1}{k}$$

(1-94) $E(X) = 3$

$E(X^2) = 13 \Rightarrow \sigma^2 = 13 - 9 = 4$

FIND BOUND ON $P_r[-2 < X < 8]$

$P_r[-2 < X < 8] = P_r[-5 < X - 3 < 5]$

$= P_r[|X - 3| < 5]$

$5 = k\sigma^2 \Rightarrow k = 5/\sigma^2 = 5/4$

$P_r[|X-3| < \frac{5}{4}\sigma^2]$

$= 1 - P_r[|X-3| \geq \frac{5}{4}|\sigma^2]$

BUT

$P_r[|X-3| \geq 5/4\,\sigma^2] \leq \frac{16}{25}$

$-P_r[|X-3| \geq 5/4\,\sigma^2] \geq -\frac{16}{25}$

$\Rightarrow P_r[|X-3| < \frac{5}{4}\sigma^2]$

$= P_r[-2 < X < 8]$

$\geq 1 - \frac{16}{25} = 9/25$

(SHOULD BE $\frac{21}{25}$)

(1-95)

$$M(t)e^{-at} = \frac{E[e^{xt}]}{e^{at}}$$

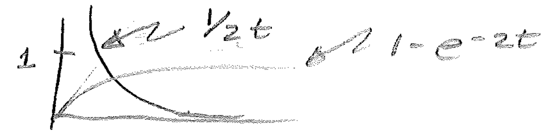
$$\geq P_r[e^{xt} \geq e^{at}]$$
$$= P_r[xt \geq at]$$
$$= \begin{cases} P_r[x \geq a] & ; t>0 \\ P_r[x \leq a] & ; t<0 \end{cases}$$

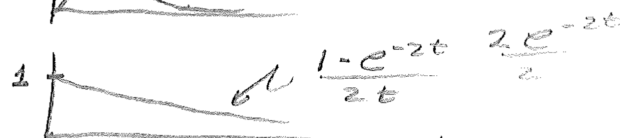
$$(1\text{-}96)\qquad M(t)=\begin{cases}\dfrac{e^{t}-e^{-t}}{2t} & ; t\neq 0\\ 1 & ; t=0\end{cases}$$

$$M(t)=E[e^{tx}]$$

(a) $P_r[X\geq 1]\leq e^{-t}M(t)\qquad ; 0<t<\infty$

$\leq \dfrac{1-e^{-2t}}{2t}$





$\Rightarrow P[X\geq 1]\leq \dfrac{1-e^{-2t}}{2t}<1\ ; 0<t<\infty$

$P_r[X\leq 1]\leq e^{-t}M(t)\qquad -\infty<t<0$

$\leq \dfrac{1-e^{-2t}}{2t}\qquad -\infty<t<1$

$P_r[X\geq 1]=1-P_r[X\leq 1]$

$=1-P_r[X<1]\geq 1-\dfrac{1-e^{-2t}}{2t}$

$\Rightarrow P_r[X\geq 1]\geq 1\quad \therefore P_r[X\geq 1]=1$

(b) $P_r[X\leq -1]\leq e^{+t}M(t)\qquad -\infty<t<0$

$=\dfrac{e^{2t}-1}{2t}\geq 1\qquad \dfrac{2te^{2t}}{2t}$

$P_r[X\leq -1]\leq e^{t}M(t)\qquad -\infty<t<0$

$\leq \dfrac{e^{2t}-1}{2t}\leq 1$

$P_r[X\leq -1]<1$

(2-1) $P(C_1) > 0$

$$P(C_2 \cup C_3 \cup C_4 \ldots | C_1)$$
$$= P[C_1 \cap (C_2 \cup C_3 \cup C_4 \cup \ldots)] / P[C_1]$$
$$= P[(C_1 \cap C_2) \cup (C_1 \cap C_3) \cup (C_1 \cap C_4) \cup \ldots] / P(C_1)$$

EACH TERM IS DISJOINT

$$= \frac{P(C_1 \cap C_2)}{P(C_1)} + \frac{P(C_1 \cap C_3)}{P(C_1)} + \ldots$$

$$= P[C_1 / C_2] + P[C_1 / C_3] + \ldots$$

(2-2)

$$P[C_1 \cap C_2 \cap C_3 \cap C_4]$$
$$= P[C_4 \cap (C_1 \cap C_2 \cap C_3)]$$
$$= P[C_4 / C_1 \cap C_2 \cap C_3]\, P[C_1 \cap C_2 \cap C_3]$$

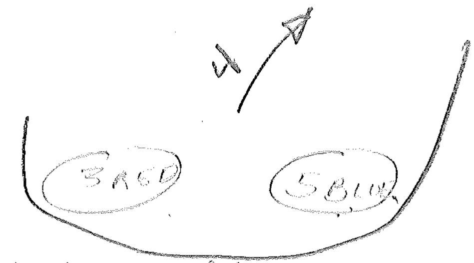
$$P[C_1 \cap C_2 \cap C_3] = P[C_3 \cap (C_1 \cap C_2)]$$
$$= P[C_3 / C_1 \cap C_2]\, P[C_1 \cap C_2]$$

$$P[C_1 \cap C_2] = P[C_2 / C_1]\, P(C_1)$$

$$\therefore P[C_1 \cap C_2 \cap C_3 \cap C_4]$$

$$= P[C_1]\, P(C_2 / C_1)\, P(C_3 / C_1 \cap C_2)\, P[C_4 / C_1 \cap C_2 \cap C_3$$

(2-3)



DRAW 4 CHIPS

(b) $P[X_4 = \text{BLUE}]$

$= P[X_1 \;\&\; X_2$ ARE RED & X_3 IS BLUE$]$

$= \cancel{\frac{3}{8}} \cdot \cancel{\frac{2}{7}} \cdot \frac{5}{\cancel{6}}$

$= 5/56$

(a) $P[$COLORS ALTERNATE$]$

$= P[BRBR] + P[RBRB]$

$= \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{4}{6} \cdot \frac{2}{5} + \frac{3}{8} \cdot \frac{5}{7} \cdot \frac{2}{6} \cdot \frac{4}{5}$

$= \cancel{2}\left[\cancel{\frac{5}{8}}\, \frac{3}{7} \cdot \cancel{\frac{4}{6}} \cdot \cancel{\frac{2}{5}}\right] = \frac{1}{7}$

(c) $X+1$ = # DRAW NEEDED FOR FIRST BLUE CHIP (w/o REPLACE)

$P[X=0] = 5/8$

$P[X=1] = \frac{3}{8} \cdot \frac{5}{7} = \frac{15}{56}$

$P[X=2] = \cancel{\frac{3}{8}} \cdot \frac{2}{7} \cdot \frac{5}{\cancel{6}} = \frac{5}{56}$

$P[X=3] = \frac{3}{8}\;\frac{2}{7}\;\frac{1}{6}\;\frac{5}{5} = \frac{1}{56}$

OR

C_1 = FIRST X CHIPS ARE RED

$P[C_1] = \binom{3}{X}\binom{5}{0} \Big/ \binom{8}{X} = \binom{3}{X} \Big/ \binom{8}{X}$

C_2 = $(X+1)^{ST}$ CHIP IS RED GIVEN FIRST X BLUE

$P(C_2/C_1) = \frac{5}{8-X}$

$P[C_2 \cap C_1] = P[$~~FIRST~~ $(X+1)^{ST}$ CHIPS ~~ARE~~ IS RED AND FIRST X BLUE$]$

$= P(C_1)\, P(C_2/C_1)$

$= \frac{5}{8-X} \; \frac{\binom{3}{X}}{\binom{8}{X}}$

(2-4)

C_1 = 2 KINGS IN HAND OF 13

C_2 = AT LEAST 3 KIGS IN HAND OF 13

FIND $P[C_2/C_1]$

OBVIOUSLY:

$$P[C_2/C_1] = \frac{\binom{2}{1}\binom{48}{10} + \binom{2}{2}\binom{48}{9}}{\binom{50}{11}}$$

(2-5)

$C =$ AT LEAST ONE MATCHING PAIR

$C^* =$ NO MATCHING PAIRS

FIND $P(C) = 1 - P(C^*)$

DENOTE DRAWS BY

$X_1, X_2, X_3, \ldots X_6$

$$P(C^*) = P[X_2 \neq X_1 / X_1]\, P[X_3 \neq X_2 \neq X_1 / X_1 X_2] \ldots P[X_6 \neq X_5 \neq X_4 \ldots / X_1 X_2 X_3 X_4]$$

$S_1 \quad S_2 \quad S_3 \quad S_4 \quad S_5 \quad S_7 \quad S_8$

$2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2$

$$P(C^*) = \left[\frac{\binom{14}{1}\binom{1}{0}}{\binom{15}{1}}\right]\left[\frac{\binom{12}{1}\binom{2}{0}}{\binom{14}{1}}\right]\left[\frac{\binom{10}{1}\binom{3}{0}}{\binom{13}{1}}\right]\left[\frac{\binom{8}{1}\binom{4}{0}}{\binom{12}{1}}\right]\left[\frac{\binom{6}{1}\binom{5}{0}}{\binom{11}{1}}\right]\left[\frac{\binom{4}{1}\binom{6}{0}}{\binom{10}{1}}\right]\left[\frac{\binom{2}{1}\binom{7}{0}}{\binom{9}{1}}\right]$$

$$= \frac{\cancel{14} \cdot \cancel{12} \cdot \cancel{10} \cdot 8 \cdot \cancel{6}^{\,2} \cdot 4 \cdot 2}{15 \cdot \cancel{14} \cdot 13 \cdot \cancel{12} \cdot 11 \cdot \cancel{10} \cdot \cancel{9}\,3}$$

$$= \frac{2 \cdot 8 \cdot 4 \cdot 2}{15 \cdot 13 \cdot 11 \cdot 3} = \frac{128}{6435}$$

$$P(C) = 1 - \frac{128}{6435} = \frac{6307}{6435}$$

~~$P(C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5 \cap C_6)$~~

~~$= P[C_1 | C_2 \cap C_3 \cap C_4 \cap C_5 \cap C_6]\, P[C_2 | C_3 \cap C_4 \ldots) \ldots P(C_6)$~~

~~$= 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11$~~ [illegible]

(2-6) 4 RED 5 WHITE 1 BLUE DRAW 3

C_1 = EACH CHIP DIFFERENT COLOR

C_2 = EXACTLY 1 RED CHIP

FIND $P(C_1/C_2)$

$$P[C_1 \cap C_2] = \frac{\binom{4}{1}\binom{5}{1}\binom{1}{1}}{\binom{10}{3}} = P(C_1)$$

$$= \frac{20}{\binom{10}{3}}$$

$$P[C_1/C_2] = \frac{P[C_1 \cap C_2]}{P[C_2]}$$

$$P[C_2] = \frac{\binom{4}{1}\binom{6}{2}}{\binom{10}{3}}$$

$$= \frac{4 \cdot \frac{\cancel{6}^{3} \cdot 5}{\cancel{2}}}{\binom{10}{3}} = \frac{60}{\binom{10}{3}}$$

$$\Rightarrow P[C_1/C_2] = \frac{20}{60} = \frac{1}{3}$$

(2-7) $C \in C_1 \cup C_2 \cup C_3 \ldots C_m$

$P(C) = P[C \cap \{C_1 \cup C_2 \ldots C_m\}]$

SINCE ALL C_i ARE DISJOINT

$P(C) = P[(C \cap C_1) \cup (C \cap C_2) \cup \ldots \cup (C \cap C_m)]$

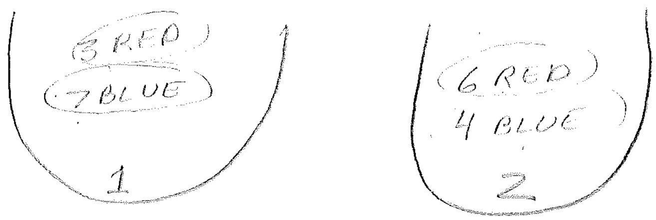
ALL OF THESE ELEMENTS ARE ALSO DISJOINT

$$\Rightarrow P(C) = P[C \cap C_1] + P[C \cap C_2) + \ldots$$
$$= P(C_1)\, P(C/C_1) + P(C_2)\, P(C/C_2) + \ldots + P[C_m]\, P[C/C_m]$$

ASSUME $P[C] > 0$

$$P[C_i/C] = P[C_i \cap C] / P(C)$$
$$= \frac{P[C/C_i]\, P[C_i]}{P[C]}$$
$$= \frac{P[C_i]\, P[C/C_i]}{P(C_1)P(C/C_1) + \ldots + P(C_m)\, P(C/C_m)}$$

(2-8)



(a) $P[\text{CHIP IS RED}] = \frac{9}{20}$

(b) R = RED CHIP

B_2 = BOWL 2

FIND $P[B_2/R]$

USE BAYES:

$$P[B_2/R] = \frac{P(B_2)\,P[R/B_2]}{P(B_2)\,P(R/B_2) + P(B_1)\,P(R/B_1)}$$

$$P[B_2/R] = \frac{\frac{1}{2}\,{}^{6}/_{10}}{\frac{1}{2}\frac{6}{10} + \frac{1}{2}\frac{3}{10}} = \frac{6}{9} = \frac{2}{3}$$

(2-9)

C_1: THE CHIP FROM BOWL II IS BLUE

C_2: 2 RED & THREE BLUE CHIPS XFERRED

$$P[C_2|C_1] = \frac{P[C_2 \cap C_1]}{P[C_1]} = \frac{P[C_1|C_2]P(C_2)}{P(C_1)}$$

$$P(C_1 | C_2)P(C_2)$$

$$= \binom{4}{3}\binom{6}{2} \Big/ \binom{10}{5} \cdot \frac{3}{5}$$

C = EVENT OF @ LEAST 1 BLUE CHIPS IS XFERRED

$$P(C_1) = \frac{\binom{4}{1}\binom{6}{4}}{\binom{10}{5}} \frac{1}{5} + \frac{\binom{4}{2}\binom{6}{3}}{\binom{10}{5}} \cdot \frac{2}{5}$$

$$+ \frac{\binom{4}{3}\binom{6}{2}}{\binom{10}{5}} \frac{3}{5} + \frac{\binom{4}{4}\binom{6}{1}}{\binom{10}{5}} \frac{4}{5}$$

$$\therefore P(C_2|C_1) = 5/14$$

(2-10) $f(x_1, x_2) = x_1 + x_2 \ ; \ 0 < x_1, x_2 < 1$

FIND $E[x_2 | x_1]$ & $var[x_2 | x_1]$

$f(x_2|x_1) = f(x_1 x_2) / f(x_1)$

$f(x_1) = \int_0^1 (x_1 + x_2)\, dx_2$

$= x_1 x_2 + \frac{1}{2} x_2^2 \Big|_0^1 = x_1 + \frac{1}{2}$

$f(x_2|x_1) = \dfrac{x_1 + x_2}{x_1 + \frac{1}{2}}$

$E[x_2|x_1] = \int_0^1 \dfrac{x_1 x_2 + x_2^2}{x + \frac{1}{2}}\, dx_2$

$= \dfrac{\frac{1}{2} x_1 x_2^2 + \frac{1}{3} x_3^3}{x_1 + \frac{1}{2}} \Big|_0^1 = \dfrac{\frac{1}{2} x_1 + \frac{1}{3}}{x_1 + \frac{1}{2}} = \dfrac{3x_1 + 2}{6x_1 + 3}$

$= \dfrac{3x_1 + 2}{3(2x_1 + 3)}$

$E[x_2^2|x_1] = \int_0^1 \dfrac{x_1 x_2^2 + x_2^3}{x_1 + \frac{1}{2}}\, dx_2$

$= \dfrac{\frac{1}{3} x_1 x_2^3 + \frac{1}{4} x_2^4}{x_1 + \frac{1}{2}} \Big|_0^1$

$= \dfrac{\frac{1}{3} x_1 + \frac{1}{4}}{x_1 + \frac{1}{2}} = \dfrac{4x_1 + 3}{6[2x_1 + 3]}$

$var[x_2|x_1] = E[x_2^2|x_1] - E^2[x_2|x_1]$

$= \dfrac{4x_1 + 3}{6(2x_1 + 3)} - \left[\dfrac{3x_1 + 1}{3(2x_1 + 3)}\right]^2$

(2-11)(a) $f(x_1 | x_2) = \dfrac{c_1 x_1}{x_2^2}$; $0 < x_1 < x_2 < 1$

$\int f(x_1|x_2)\,dx_1$

$= \int_0^{x_2} \dfrac{c_1 x_1}{x_2^2}\,dx_1 = \dfrac{c_1}{2} = 1 \Rightarrow c_1 = 2$

$f_2(x_2) = c_2 x_2^4$; $0 < x_2 < 1$

$= \int_0^1 c_2 x_2^4\,dx = \dfrac{c_2}{5} \Rightarrow c_2 = 5$

(b) $f(x_1, x_2) = f(x_1|x_2)\,f_2(x_2)$

$= \dfrac{2x_1}{x_2^2}\, c_2 x_2^4$ [illegible]

$= 10 x_1 x_2^2$; $0 < x_1 < x_2 < 1$ [illegible]

(c) $Pr[\frac{1}{4} < x_1 < \frac{1}{2} | x_2$ [illegible]]

$= \int_{1/4}^{1/2} 2 \dfrac{x_1}{x_2^2}\,dx_1$ [illegible] $\frac{1}{2} x_1^2 \Big|_{1/4}^{1/2}$

$=$ [illegible]

$= \dfrac{64}{25}\left[\frac{1}{4} - \frac{1}{16}\right]$ [illegible]

$=$ [illegible]

(d) $Pr[\frac{1}{4} < x_1 < \frac{1}{2}]$

$f_1(x_1) = \int_{x_1}^1 10 x_1 x_2^2\,dx_2$

$= 10 x_1 \frac{1}{3} x_2^3 \Big|_{x_1}^1$

$= \frac{10}{3} x_1 [1 - x_1^3]$

$= \frac{10}{3}$ [illegible]

$Pr[\frac{1}{4} < x_1 < \frac{1}{2}] = \frac{10}{3}\int_{1/4}^{1/2} [x_1 - x_1^4]\,dx_1$

$= \frac{10}{3}\left[\frac{1}{2} x_1^2 - \frac{1}{5} x_1^5\right]_{1/4}^{1/2}$

(2-12) $f(x_1, x_2) = 21 x_1^2 x_2^3 \quad ; \quad 0 < x_1 < x_2 < 1$

FIND $E[X_1 | X_2]$ $\quad Var[X_1 | X_2]$

$f(x_1 | x_2) = f(x_1, x_2) / f_2(x_2)$

$f_2(x_2) = 21 x_2^3 \int_0^{x_2} x_1^2 \, dx_1$

$= 7 x_2^3 x_2^3$

$= 7 x_2^6$

$\Rightarrow f(x_1 | x_2) = \frac{21 x_1^2 x_2^3}{7 x_2^6}$

$= 3 x_1^2 / x_2^3$

$E[X_1 | X_2] = \frac{3}{x_2^3} \int_0^{x_2} x_1^3 \, dx_1$

$= \frac{3}{x_2^3} \frac{1}{4} x_1^4 \Big|_0^{x_2} = 3/4 \, X_2$

$E[X_1^2 | X_2] = 3/x_2^3 \int_0^{x_2} x_1^4 \, dx_1$

$= \frac{3}{x_2^3} \frac{1}{5} x_1^5 \Big|_0^{x_2}$

$= \frac{3}{5} x_2^2$

$\Rightarrow Var[X_1^2 | X_2] = \frac{3}{5} x_2^2 - 9/16 \, X_2^2$

$= \frac{48 - 45}{80} X_2^2$

$= \frac{3}{80} X_2^2$

$$f(x_1,x_2)=\frac{x_1+2x_2}{18}$$

$(x_1,x_2) = (1,1)\ (1,2)\ (2,1)\ (2,2)$

Find $E[X_2 \mid X_1]$

$$f(x_2 \mid x_1) = f(x_1,x_2)/f(x_1)$$

$$f(x_1) = \sum_{x_2} f(x_1,x_2) = \sum_{x_2} \frac{x_1+2x_2}{18}$$

$$= \frac{x_1}{9} + \frac{1}{9}\sum_{x_2} x_2 = \frac{x_1}{9} + \frac{1}{9}(1+2) [illegible]$$

$$= \frac{4x_1+12}{18}$$

$$\Rightarrow f(x_2 \mid x_1) = \frac{x_1+2x_2}{4x_1+12}$$

$$E[X_2 \mid X_1] = \sum_{x_2} \frac{x_1 x_2 + 2x_2^2}{4x_1+12}$$

$$= \frac{1}{4x_1+12}\left[6x_1 + 2(1+ [illegible])\right]$$

$$= [illegible]$$

$$E[X_2^2 \mid X_1] = \frac{2x_1+6}{4x_1+12}[illegible]$$

$$Var[X_2 \mid X_1] = [illegible] - \left(\frac{3x_1+10}{2x_1+6}\right)^2$$

2.14

(a) $p(x_1, x_2, x_3) = \dfrac{\binom{13}{x_1}\binom{13}{x_2}\binom{13}{x_3}\binom{13}{5-x_1-x_2-x_3}}{\binom{52}{5}}$, $x_i = 0, \dots, 4$

(b) $p_1(x_1) = \binom{13}{x_1}\binom{39}{5-x_1} \Big/ \binom{52}{5}$

(c) $p(x_2, x_3 \mid x_1 = 3)$

$$= \frac{p(3, x_2, x_3)}{p_1(3)}$$

$$= \frac{\binom{13}{x_2}\binom{13}{x_3}\binom{13}{2-x_2-x_3}}{\binom{39}{2}}$$

MARGINALS:

$f_1(x_1) = \sum_{x_2} f(x_1, x_2)$

$$= \begin{cases} \frac{1}{18} + \frac{3}{18} & ; x_1 = 0 \\ \frac{4}{18} + \frac{3}{18} & ; x_1 = 1 \\ \frac{6}{18} + \frac{1}{18} & ; x_1 = 2 \end{cases}$$

$$= \begin{cases} 4/18 & ; x_1 = 0 \\ [illegible] & ; x_1 = 1 \\ [illegible]/18 & ; x_1 = 2 \end{cases}$$

$f_2(x_2) = [illegible]$; $x_2 = 0$

$[illegible]$; $x_2 = 1$

$$= \begin{cases} [illegible]/18 & ; x_2 = 0 \\ [illegible] & ; x_2 = 1 \end{cases}$$

$E[X_1 | X_2] = \sum x_1 f(x_1 | x_2)$

$f(x_1 | x_2)$ = [illegible]

$f(x_1 \mid x_2)$	(x_1, x_2)
$\frac{1}{11}$	(0,0)
$\frac{3}{7}$	(0,1)
$\frac{4}{11}$	(1,0)
$\frac{3}{7}$	(1,1)
$\frac{6}{11}$	(2,0)
$\frac{1}{7}$	(2,1)

$$E[X_1 | X_2] = \begin{cases} 0 \cdot \frac{1}{11} + 1 \cdot \frac{4}{11} + 2 \cdot \frac{6}{11} & ; x_2 = 0 \\ 0 \cdot \frac{3}{7} + 1 \cdot \frac{3}{7} + 2 \cdot \frac{1}{7} & ; x_2 = 1 \end{cases}$$

$$= \begin{cases} \frac{16}{11} & ; x_2 = 0 \\ \frac{5}{7} & ; x_2 = 1 \end{cases}$$

$f(x_2 | x_1) = f(x_1, x_2) / f(x_1) =$

$f(x_2 \mid x_1)$	(x_1, x_2)
$\frac{1}{4}$	(0,0)
$\frac{3}{4}$	(0,1)
$\frac{4}{7}$	(1,0)
$\frac{3}{7}$	(1,1)
$\frac{6}{7}$	(2,0)
$\frac{1}{7}$	(2,1)

$$E[X_2 | X_1] = \begin{cases} 0 \cdot \frac{1}{4} + 1 \cdot \frac{3}{4} & ; x_1 = 0 \\ 0 \cdot \frac{4}{7} + 1 \cdot \frac{3}{7} & ; x_1 = 1 \\ 0 \cdot \frac{6}{7} + 1 \cdot \frac{1}{7} & ; x_1 = 2 \end{cases}$$

$$= \begin{cases} 3/4 & ; x_1 = 0 \\ 3/7 & ; x_1 = 1 \\ 1/7 & ; x_1 = 2 \end{cases}$$

(2-17)

$$f(x \mid X > x_0) = \frac{f(x)}{1 - F(x_0)}, \quad x > x_0$$

(a) $\int_{x_0}^{\infty} f(x \mid X > x_0)\,dx$

$$= \frac{1}{1-F(x_0)} \int_{x_0}^{\infty} f(x)\,dx$$

$$= \frac{1}{1-F(x_0)} P[X > x_0]$$

$$= \frac{1}{1-F(x_0)} \left[1 - P[X \le x_0]\right]$$

$$= \frac{1}{1-F(x_0)} (1 - F(x_0)) = 1$$

[illegible] $f(x \mid X > x_0) \geq 0$

(b) $f(x) = e^{-x}, \quad 0 < x \leq \infty$

$P_r[X > 2 \mid X > 1]$

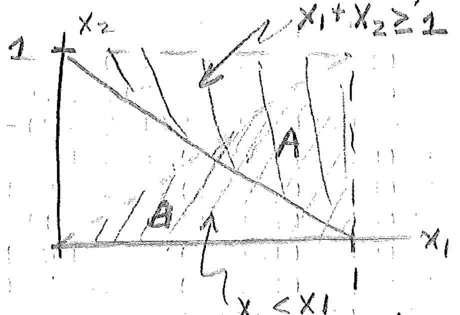
$$= \frac{P[X > 2, X > 1]}{P[X > 1]}$$

$$= \frac{P[X > 2]}{P[X > 1]} = \frac{e^{-2}}{e^{-1}} = e^{-1}$$

(2-16) (a) $f(x_1) = 1 \quad ; 0 < x_1 < 1$

$$f(x_2 | x_1) = \frac{1}{x_1} \quad ; 0 < x_2 < x_1$$

(b) $f(x_1, x_2) = \frac{1}{x_1} \quad ; 0 < x_2 < x_1 < 1$



FIND

$Pr[x_1 + x_2 \geq 1]$

$$Pr[X_1 + X_2 \geq 1] = \int_A f(x_1, x_2)\, dx$$

$$= 1 - Pr[X_1 + X_2 < 1]$$

$$Pr[X_1 + X_2 < 1] = \int_0^{1/2} \frac{1}{x_1} \int_0^{x_1} dx_2\, dx_1 + \int_{\frac{1}{2}}^1 \frac{1}{x_1} \int_0^{(1-x_1)} dx_2\, dx_1$$

$$= \int_0^{1/2} dx_1 + \int_{1/2}^1 \left(\frac{1}{x_1} - 1\right) dx_1$$

$$= \frac{1}{2} + \left[\ln x_1 - x_1\right]_{1/2}^1$$

$$= \frac{1}{2} + \left[(-1) - \left(\ln \frac{1}{2} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} - 1 + \ln 2 + \frac{1}{2} = \ln 2$$

$$\Rightarrow Pr[X_1 + X_2 \geq 1] = 1 - \ln 2 \simeq 0.307$$

(c) FIND $E(x_1 | x_2)$

$$f(x_2) = \int_{x_1} \frac{1}{x_1}\, dx_1 = \int_{x_2}^1 \frac{1}{x_1}\, dx_1$$

$$= \ln(x_1) \Big|_{x_2}^1 = -\ln x_2$$

$$\Rightarrow f(x_1 / x_2) = f(x_1, x_2) \big/ -\ln x_2$$

$$= \frac{-1}{x_1 \ln x_2} \qquad 0 < x_2 < x_1 < 1,$$

$$E[x_1 | x_2] = \int_{x_1} \frac{-1}{\ln x_2}\, dx_1$$

$$= \frac{-1}{\ln x_2} \int_{x_2}^1 dx_1 = \frac{x_2 - 1}{\ln x_2}$$

(2-18) (a) $f(x,y) = \frac{1}{3}$, $(x,y) = (0,0), (1,1), (2,2)$

$f(x) = \frac{1}{3}$; $x = 0, 1, 2$

$f(y) = \frac{1}{3}$; $y = 0, 1, 2$

$\mu_x = \frac{1}{3}(0) + \frac{1}{3}(1) + \frac{1}{3}(2) = 1 = \mu_y$

$E(X^2) = \frac{1}{3}0^2 + \frac{1}{3}(1)^2 + \frac{1}{3}(2)^2 = \frac{5}{3} = E(Y^2)$

$E(XY) = \frac{1}{3}0 \cdot 0 + \frac{1}{3}(1 \cdot 1) + \frac{1}{3}(2 \cdot 2) = 5/3$

$\rho_{12} = \dfrac{E[XY] - \mu_1\mu_2}{\sigma_x \sigma_y}$ $\quad \cancel{\dfrac{5/3 - 1}{5/3} = \dfrac{2}{5}}$

$\sigma_x^2 = 5/3 - 1 = 2/3 = \sigma_y^2$

$\Rightarrow \rho_{12} = \dfrac{5/3 - 1}{2/3} = 1$

$cov = 5/3 - 1 = 2/3$

(b) $f(x,y) = \frac{1}{3}$, $(x,y) = (0,2), (1,1) (2,0)$

$f(x) = \frac{1}{3}$; $x = 0, 1, 2$

$\Rightarrow \mu_x = 1$

$f(y) = \frac{1}{3}$, $y = 0, 1, 2 \Rightarrow \mu_y = 1$

$E(X^2) = \frac{1}{3}0^2 + \frac{1}{3}1^2 + \frac{1}{3}2^2 = 5/3 = E(Y^2)$

$\sigma_1^2 = \sigma_2^2 = 5/3 - 1 = 2/3$

$E(X,Y) = \frac{1}{3}(0 \cdot 2) + \frac{1}{3}(1 \cdot 1) + \frac{1}{3}(2 \cdot 0) = \frac{1}{3}$

$\rho_{12} = \dfrac{\frac{1}{3} - 1}{2/3} = -1$

$cov = -2/3$

(c) $f(x,y) = \frac{1}{3}, \quad (x,y) = (0,0), (1,1), (2,0)$

$f(x) = \frac{1}{3} \quad ; \quad x = 0, 1, 2 \Rightarrow \mu_x = 1$

$E[X^2] = 0^2 \cdot \frac{1}{3} + 1^2 \frac{1}{3} + 2^2 \frac{1}{3} = 5/3$

$\Rightarrow \sigma_x^2 = 2/3$

$$f(y) = \begin{cases} 2/3 & ; \; y = 0 \\ 1/3 & ; \; y = 1 \end{cases}$$

$\mu_y = \frac{2}{3}(0) + \frac{1}{3}(1) = \frac{1}{3}$

$E(Y^2) = \frac{2}{3} 0^2 + \frac{1}{3}(1)^2 = \frac{1}{3}$

$\sigma_y^2 = \frac{1}{3} - \frac{1}{9} = 2/9$

$E(XY) = \frac{1}{3}(1 \cdot 1) = \frac{1}{3}$

$\Rightarrow cov = \frac{1}{3} - \left(\frac{1}{3}\right)(1) = 0$

$\Rightarrow \rho_{12} = 0$

(2-19) $f(x,y) = \begin{cases} \frac{2}{15} & (1,1) \\ \frac{4}{15} & (1,2) \\ \frac{3}{15} & (1,3) \\ \frac{1}{15} & (2,1) \\ \frac{1}{15} & (2,2) \\ \frac{4}{15} & (2,3) \end{cases}$

$$E(XY) = \frac{2}{15}(1) + \frac{4}{15}2 + \frac{3}{15}3 + \frac{1}{15}2 + \frac{1}{15}4 + \frac{4}{15}6$$
$$= \frac{1}{15}\left[2+8+9+2+4+24\right] = \frac{49}{15}$$

$$f(x) = \begin{cases} 9/15 & ; x = 1 \\ 6/15 & x = 2 \end{cases}$$

$$\mu_x = \frac{9}{15}\cdot 1 + \frac{6}{15}\cdot 2 = \frac{1}{15}(9+12) = \frac{21}{15} = \frac{7}{5}$$
$$E[X^2] = (1)^2\frac{9}{15} + 2^2\frac{6}{15} = \frac{1}{15}[9+24] = \frac{33}{15} = \frac{11}{5}$$
$$\sigma_x^2 = \frac{11}{5} - \frac{49}{25} = \frac{1}{25}[55-49] = 6/25$$

$$f(Y) = \begin{cases} 3/15 & Y = 1 \\ 5/15 & Y = 2 \\ 7/15 & Y = 3 \end{cases}$$

$$\mu_Y = \frac{3}{15} + 2\frac{5}{15} + \frac{3\cdot 7}{15} = \frac{3+10+21}{15} = \frac{34}{15}$$
$$E(Y^2) = \frac{3}{15} + \frac{5}{15}4 + \frac{7}{15}\cdot 9 = \frac{3+20+63}{15} = \frac{86}{15}$$
$$\sigma_x^2 = \frac{86}{15} - \frac{1156}{225} = \frac{134}{225} \Rightarrow \sigma = \frac{\sqrt{134}}{15}$$

$$cov = \frac{49}{15} - \frac{7}{5}\cdot\frac{34}{15} = \frac{245-238}{75} = \frac{7}{75}$$

$$\rho = \frac{cov}{\sigma_x\sigma_y} = \frac{7/\cancel{75}}{\frac{\sqrt{6}}{\cancel{5}}\,\frac{\sqrt{134}}{\cancel{15}}} = \frac{7}{\sqrt{804}}$$

(2-20) $f(x,y) = 2, \quad 0 < x < y < 1$

(a) $f(x) = \int_Y f(x,y)\,dy$

$= 2\int_x^1 dy = 2(1-x)$

$f(y) = \int_x f(x,y)\,dx$

$= 2\int_0^y dx = 2y$

$f(x/y) = \dfrac{f(x,y)}{f(y)} = \dfrac{1}{y}$

$E[X/Y] = \dfrac{1}{y}\int_0^y x\,dx = \dfrac{1}{y}\,\dfrac{y^2}{2} = \dfrac{y}{2} \quad ; 0<y<1$

$f(y/x) = \dfrac{f(x,y)}{f(x)} = \dfrac{1}{(1-x)}$

$E[Y/X] = \dfrac{1}{1-x}\int_x^1 y\,dy = 2\dfrac{1-x^2}{(1-x)} = \dfrac{x+1}{2}$

(b) $\mu_x = 2\int_0^1 (x-x^2)\,dx = 2\left[\dfrac{x^2}{2} - \dfrac{x^3}{3}\right]$

$= 2\left[\dfrac{1}{2} - \dfrac{1}{3}\right] = 2\,\dfrac{1}{6} = \dfrac{1}{3}$

$\mu_y = \int_0^1 2y^2\,dy = \dfrac{2}{3}y^3\Big|_0^1 = 2/3$

$E(x^2) = 2\int_0^1 (x^2 - x^3)\,dx = 2\left[\dfrac{x^3}{3} - \dfrac{x^4}{4}\right] = 2\left[\dfrac{1}{3} - \dfrac{1}{4}\right]$

$= 2\,\dfrac{1}{12} = \dfrac{1}{6}$

$\Rightarrow \sigma_x^2 = \dfrac{1}{6} - \dfrac{1}{9} = \dfrac{1}{18}$

$E(y^2) = 2\int_0^1 y^3\,dy = \dfrac{1}{2}$

$\Rightarrow \sigma_y^2 = \dfrac{1}{2} - \dfrac{4}{9} = \dfrac{9-8}{18} = \dfrac{1}{18}$

$E(x,y) = 2\int_0^1 y\int_0^y x\,dx\,dy$

$= \int_0^1 y^3\,dy = \dfrac{1}{4}y^4\Big|_0^1 = \dfrac{1}{4}$

$\rho_1 = \dfrac{\frac{1}{4} - \frac{1}{3}\,\frac{2}{3}}{1/18} = 18\left[\dfrac{1}{4} - \dfrac{2}{9}\right]$

$= 18\,\dfrac{9-8}{36} = \dfrac{18}{36} = \dfrac{1}{2}$

(2-21)

$$f(x|y) = \frac{1}{y} \quad ; \quad 0 < x < y < 1$$

$$E[x^2|y] = \frac{1}{y}\int_0^y x^2\,dx$$

$$= \frac{1}{3y}\, y^3 = \frac{y^2}{3}$$

$$\Rightarrow var(x|y) = \frac{y^2}{3} - \frac{y^2}{4} = \frac{y^2}{12}$$

$$f(y|x) = \frac{1}{1-x}$$

$$E[y^2|x] = \frac{1}{1-x}\int_x^1 y^2\,dy$$

$$= \frac{1}{3}\,\frac{1}{1-x}\,(1-x^3)$$

$$= \frac{1-x^3}{3(1-x)}$$

$$\Rightarrow var(x|y) = \frac{x^3-1}{3(x-1)} - \frac{(x+1)^2}{4}$$

$$= \frac{1}{12(x-1)}\left[4(x^3-1) - 3(x+1)^2(x-1)\right]$$

$$= \frac{1}{12(x-1)}\left[4x^3 - 4 - 3(x-1)(x^2+2x+1)\right]$$

$$= \frac{1}{12(x-1)}\left[4x^3 - 4 - 3x^3 - 6x^2 - 3x + 3x^2 + 6x + 3\right]$$

$$= \frac{1}{12(x-1)}\left[x^3 - 3x^2 - 3x - 1\right]$$

$$\begin{array}{r|l}
 & x^2 + 4x + 1 \\
\hline
x-1 & x^3 + 3x^2 - 3x - 1 \\
 & x^3 - x^2 \\
\hline
 & 4x^2 - 3x - 1 \\
 & 4x^2 - 4x \\
\hline
 & x - 1
\end{array}$$

$$\Rightarrow var(x|y) = \frac{(x-1)^2}{12}$$

(2-22)

$$M(t_1, t_2) = \frac{1}{(1-t_1-t_2)(1-t_2)} \quad \begin{cases} t_2 < 1 \\ t_1+t_2 > 1 \end{cases}$$

$$\frac{\delta M}{\delta t_1} = \frac{+(1-t_2)}{(1-t_1-t_2)^2(1-t_2)^2}$$

$$= \frac{1}{(1-t_1-t_2)^2(1-t_2)}$$

$$\frac{\delta M}{\delta t}(0,0) = 1 = \mu_1$$

$$\frac{\delta M}{\delta t_2} = \frac{+[+(1-t_2)+(1-t_1-t_2)]}{(1-t_1-t_2)^2(1-t_2)^2}$$

$$= \frac{2-2t_2-t_1}{(1-t_1-t_2)^2(1-t_2)^2}$$

$$\frac{\delta M}{\delta t_2}(0,0) = 2 = \mu_2$$

$$\frac{\delta^2 M}{\delta t_1^2} = \frac{-[\cancel{-(1-t_1-t_2)^2} - 2(1-t_2)(1-t_1-t_2)]}{(1-t_1-t_2)^4(1-t_2)^2}$$

$$E[X_1^2] = \delta^2 M/\delta t_1^2(0,0)$$

$$= -[\quad -2]/1 = 2$$

$$\Rightarrow \sigma_1^2 = 2-1 = 1.$$

$$\frac{\delta^2 M}{d t_2^2} = \frac{\begin{array}{l}[(1-t_1-t_2)^2(1-t_2)^2(-2) \\ \quad -(2-2t_2-t_1)[2(t_2+t_1-1)(1-t_2)^2 \\ \qquad + (1-t_1-t_2)^2 2(t_2-1)]\end{array}}{(1-t_1-t_2)^4(1-t_2)^4}$$

$$E[X_2^2] = \frac{\delta^2 M}{\delta t_2^2}(0,0)$$

$$= \frac{-2 + 2[2+2]}{1} \cdot \frac{1}{} = 6$$

$$\Rightarrow \sigma_2^2 = 6-4 = 2$$

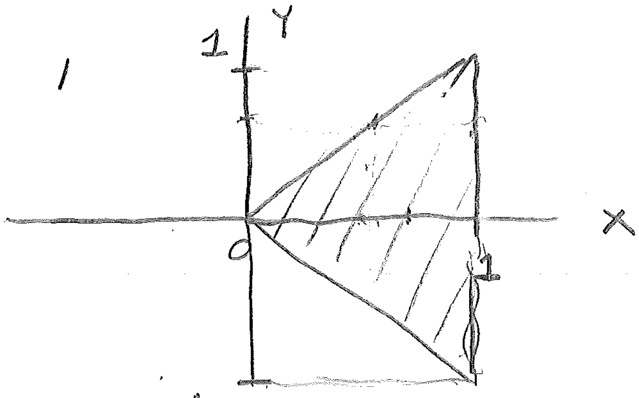
(2.23)

$$0 < |Y| < X < 1$$

$$f(x,Y) = 1 \quad ; \quad -X < Y < X < 1$$

$$0 < X < 1$$

$$|Y| < X \Rightarrow X^2 \geq Y^2$$



$$f(x) = \int_Y (1)\,dY = \int_{-X}^{X} dY = 2X \quad ; 0<X<1$$

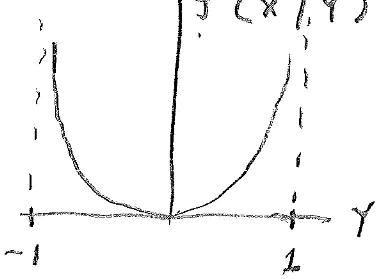
$$f(Y/X) = \frac{1}{2X} \quad ; -X<Y<X \quad 0<X<1$$

$$E[Y|X] = \frac{1}{2X}\int_Y Y\,dY = \frac{1}{2X}\int_{-X}^{X} Y\,dY$$

$$= \frac{1}{4X}\, Y^2\Big|_{-X}^{X} = \frac{1}{4X}(X^2 - X^2) = 0$$

$$f(Y) = \int_X dx = \int_{|Y|}^{1} dx = 1-|Y| \quad ;$$

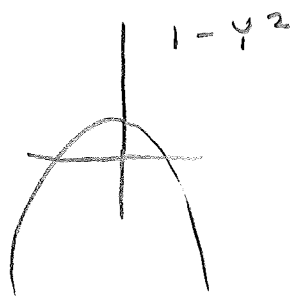
$$f(x|Y) = \frac{1}{1-|Y|} \quad ; \ 0<|Y|<X<1$$



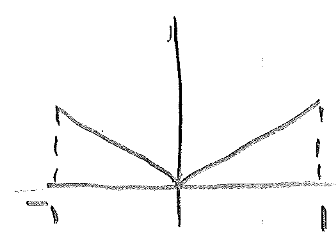
$$E[X|Y] = \frac{1}{1-|Y|}\int_{|Y|}^{1} x\,dx$$

$$= \frac{1-|Y|^2}{2(1-|Y|)}$$

$$= \frac{1-Y^2}{2(1-|Y|)} \quad -1<Y<1$$



$$= \begin{cases} \dfrac{1-Y^2}{2(1-Y)} = \dfrac{1+Y}{2} & ; Y>0 \\ \dfrac{1-Y^2}{2(1+Y)} = \dfrac{1-Y}{2} & ; Y<0 \end{cases}$$



(2-24)

$$E\left[\{(X-\mu_1)+v(Y-\mu_2)\}^2\right] \geq 0$$

$$v^2 E\left[(Y-\mu_2)^2\right] + 2v\, E\left[(X-\mu_1)(Y-\mu_2)\right] + E\left[(X-\mu_1)^2\right] \geq 0$$

LET

$$v = -\frac{E\left[(X-\mu_1)(Y-\mu_2)\right]}{E\left[(Y-\mu_2)^2\right]}$$

$$\frac{\left(E\left[(X-\mu_1)(Y-\mu_2)\right]\right)^2}{E\left[(Y-\mu_2)^2\right]} - \frac{2\left(E\left[(X-\mu_1)(Y-\mu_2)\right]\right)^2}{E\left[(Y-\mu_2)^2\right]} + E\left[(X-\mu_1)^2\right] \geq 0$$

$$\frac{\left[E\left[(X-\mu_1)(Y-\mu_2)\right]\right]^2}{E\left[(Y-\mu_2)^2\right]} \leq E\left[(X-\mu_1)^2\right]$$

$$\left[\frac{E\left[(X-\mu_1)(Y-\mu_2)\right]}{E\left[(X-\mu_1)^2\right]}\right]^2 \leq 1$$

$$\rho_{12}^2 \leq 1$$

$$\Rightarrow \quad -1 \leq \rho_{12} \leq 1$$

(2-25)

$$\psi(t_1,t_2) = \ln M(t_1,t_2)$$
$$= \ln E\left[e^{t_1x_1+t_2x_2}\right]$$
$$= \ln \int_{-\infty}^{\infty}\int f(x_1x_2)\, e^{t_1x_1+t_2x_2}\, dx_1 dx_2$$

$$\frac{\delta\psi}{\delta t_1} = \frac{\int_{-\infty}^{\infty}\int f(x_1x_2)\, x_1\, e^{t_1x_1+t_2x_2}\, dx_1dx_2}{\int_{-\infty}^{\infty}\int f(x_1x_2)\, e^{t_1x_1+t_2x_2}\, dx_1 dx_2}$$

$$\frac{\delta\psi(0,0)}{\delta t_1} = \frac{\int_{-\infty}^{\infty}\int f(x_1x_2)\, x_1\, dx_1\, dx_2}{\cancel{\iint f(x_1x_2)\, dx_1 dx_2}}$$
$$= \int_{-\infty}^{\infty} f_1(x_1)\, x_1\, dx_1 = E(x_1)$$

$$\frac{\delta^2\psi}{\delta t_1^2} = \frac{\left[\iint f(x_1x_2)\, e^{t_1x_1+t_2x_2}\, dx_1dx_2\right] \times \left[\iint f(x_1x_2)\, x_1^2\, e^{t_1x_1+t_2x_2}\, dx_1dx_2\right] - \left[\iint f(x_1x_2)\, x_1\, e^{t_1x_1+t_2x_2}\, dx_1dx_2\right] \times \left[\iint f(x_1x_2)\, x_1\, e^{t_1x_1+t_2x_2}\, dx_1dx_2\right]}{\left[\iint f(x_1x_2)\, e^{t_1x_1+t_2x_2}\, dx_1dx_2\right]^2}$$

$$\frac{\delta^2\psi(0,0)}{\delta t_1^2} = \frac{\iint f(x_1x_2)\, x_1^2\, dx_1 dx_2 - \left[\iint f(x_1x_2)\, x_1\, dx_1 dx_2\right]^2}{(1)^2}$$
$$= \int x_1^2 f_1(x_1)\, dx_1 - \left[\int x_1 f(x_1)\, dx_1\right]^2$$
$$= E[x_1^2] - E^2[x_1] = \sigma_1^2$$

$$\frac{\delta^2\psi}{\delta t_1\,\delta t_2}= \frac{\begin{array}{l}\left[\iint f(x_1,x_2)\,e^{t_1x_1+t_2x_2}\,dx_1\,dx_2\right]\\ \times\left[\iint f(x_1,x_2)\,x_1x_2\,e^{t_1x_1+t_2x_2}\,dx_1dx_2\right]\\ -\left[\iint f(x_1,x_2)\,x_1\,e^{t_1x_1+t_2x_2}\,dx_1dx_2\right]\\ \times\left[\iint f(x_1,x_2)\,x_2\,e^{t_1x_1+t_2x_2}\,dx_1dx_2\right]\end{array}}{\left[\iint f(x_1,x_2)\,e^{t_1x_1+t_2x_2}\,dx_1dx_2\right]^2}$$

$$\frac{\delta^2\psi(0,0)}{\delta t_1\,\delta t_2}= \frac{\begin{array}{l}\iint f(x_1,x_2)\,dx_1dx_2\times\iint x_1x_2\,f(x_1,x_2)\,dx_1dx_2\\ -\iint f(x_1,x_2)\,x_1\,dx_1dx_2\times\iint f(x_1,x_2)\,x_2\,dx_1dx_2\end{array}}{(1)^2}$$

$$= E[X_1X_2]-\int f_1(x_1)\,x_1\,dx_1\int f_2(x_2)\,x_2\,dx_2$$

$$= E[X_1X_2]-\mu_1\mu_2$$

$$= \mathrm{Cov}(X_1,X_2)$$

(2-26) $\mu_1\ \mu_2\ \mu_3\quad \sigma_1\ \sigma_2\ \sigma_3\ \rho_{12}\ \rho_{13}\ \rho_{23}$

$$E[x_1-\mu_1 \mid x_2, x_3] = b_2(x_2-\mu_2)+b_3(x_3-\mu_3)$$

$$E[x_1-\mu_1 \mid x_2, x_3] = \int_{x_1} (x_1-\mu_1)\, f(x_1 \mid x_2 x_3)\, dx_1$$

$$= \int_{x_1} (x_1-\mu_1) \frac{f(x_1 x_2 x_3)}{f(x_2 x_3)}\, dx_1$$

$$= \frac{\int_{x_1}(x_1-\mu_1)\, f(x_1 x_2 x_3)\, dx_1}{\int_{x_1} f(x_1 x_2 x_3)\, dx_1} = b_2(x_2-\mu_2)+b_3(x_3-\mu_3)$$

$$\int_{x_1}(x_1-\mu_1)\, f(x_1 x_2 x_3)\, dx_1 = \int_{x_1}\left[b_2(x_2-\mu_2)+b_3(x_3-\mu_3)\right] f(x_1 x_2 x_3)\, dx_1$$

$$\iiint (x_1-\mu_1)\, f(x_1 x_2 x_3)\, dx_1\, dx_2\, dx_3 = \iiint \left[b_2(x_2-\mu_2)+b_3(x_3-\mu_3)\right] f(x_1 x_2 x_3)\, dx_1 dx_2 dx_3$$

$$E[x_1-\mu_1] = b_2 E[x_2-\mu_2] + b_3 E[x_3-\mu_3]$$

$$0 = 0$$

(YECH!)

BACK TO

$$\int_{x_1}(x_1-\mu_1)\, f(x_1 x_2 x_3)\, dx = \left[b_2(x_2-\mu_2)+b_3(x_3-\mu_3)\right] \times \int f(x_1 x_2 x_3)\, dx_1$$

DONE!

(2-27) $f(x_1, x_2) = 12 x_1 x_2 (1 - x_2)$; $0 < x_1, x_2 <$

$g(x_1) = 12 x_1$; $0 < x_1 < 1$

$h(x_2) = x_2(1 - x_2)$; $0 < x_1 < 1$

$g(x_1) > 0$ FOR $0 < x_1 < 1$

$h(x_2) > 0$ " $0 < x_2 < 1$

THUS, BY THEM 1 ON p. 77, X_1 & X_2 ARE S.I.

(2-28)

$$f(x_1, x_2) = 2e^{-x_1-x_2}\ ;\ 0<x_1<x_2<\infty$$

$$f_1(x_1) = 2e^{-x_1}\int_{x_1}^{\infty} e^{-x_2}$$

$$= -2\ e^{-x_1}\ e^{-x_2}\Big|_{x_1}^{\infty}$$

$$= -2e^{-x_1}\left[-e^{-x_1}\right]$$

$$= 2e^{-2x}$$

$$f_2(x_2) = 2e^{-x_2}\int_0^{x_2} e^{-x_1}dx_1$$

$$= -2e^{-x_2}\ e^{-x_1}\Big|_0^{x_2}$$

$$= -2e^{-x}\left[e^{-x_2}-1\right]$$

OBVIOUSLY

$$f(x_1 x_2) \neq f_1(x_1)\, f_2(x_2)$$

(2-29)

$$f(x_1, x_2) = \frac{1}{16} \qquad x_1, x_2 = 1, 2, 3, 4$$

$$f_1(x_1) = \Sigma_{x_2} f(x_1\ x_2)$$

$$= \frac{1}{4} \qquad ; x_1 = 1, 2, 3, 4$$

$$f_2(x_2) = \frac{1}{4} \qquad ; x_2 = 1, 2, 3, 4$$

$$\Rightarrow f(x_1, x_2) = f_1(x_1) f_2(x_2)$$

(2-30) X_1 & X_2 ARE S.I. BY THEM 1:

$$g(x_1) = 4x_1$$
$$h(x_2) = (1 - x_2)$$

$\Rightarrow$ BY THEM 2:

$$P_r\left[0 < x_1 < \tfrac{1}{3},\ 0 < x_2 < \tfrac{1}{3}\right]$$
$$= P_r\left[0 < x_1 < \tfrac{1}{3}\right] P_r\left[0 < x_2 < \tfrac{1}{3}\right]$$

OBVIOUSLY, SINCE $f(x_1,x_2) = 4x_1(1-x_2)$

$$f_1(x_1) = 2x_1 \quad ; 0 < x_1 < 1$$

$$\Rightarrow P_r\left[0 < x_1 < \tfrac{1}{3}\right]$$
$$= 2\int_0^{1/3} x_1\,dx_1$$
$$= x_1^2\Big|_0^{\frac{1}{3}} = \frac{1}{9}$$

$$f_2(x_2) = 2(1 - x_2)$$

$$\Rightarrow P_r\left[0 < x_2 < \tfrac{1}{3}\right]$$
$$= 2\int_0^{1/3} (1 - x_2)\,dx_2$$
$$= 2\left[x_2 - \frac{x_2^2}{2}\right]\Big|_0^{\frac{1}{3}}$$
$$= 2x_2 - x_2^2\Big|_0^{1/3}$$
$$= \frac{2}{3} - \frac{1}{9} = 5/9$$

$$\Rightarrow P_r\left[0 < x_1 < \tfrac{1}{3},\ 0 < x_2 < \tfrac{1}{3}\right]$$

$$= \frac{1}{9} \cdot \frac{5}{9} = 5/81$$

(2-31)

$$P_r\left[\{a < x_1 < b,\ -\infty < x_2 < \infty\} \cup \{-\infty < x_1 < \infty,\ c < x_2 < d\}\right]$$

x_1 & x_2 ARE IND

$$= P_r\left[\{a < x_1 < b,\ -\infty < x_2 < \infty\}\right] + P_r\left[-\infty < x_1 < \infty,\ c < x_2 < d\right] - P_r\left[\{a < x_1 < b,\ -\infty < x_2 < \infty\} \cap \{-\infty < x_1 < \infty,\ c < x_2 < d\}\right]$$

$$= P_r[a < x_1 < b] + P_r[c < x_2 < d] - P_r\left[\{a < x_1 < b\} \cap \{c < x_2 < d\}\right]$$

$$= \frac{2}{3} + \frac{5}{8} - \frac{2}{3}\cdot\frac{5}{8}$$

$$= \frac{16+15-10}{24} = \frac{21}{24} = 7/8$$

(2-32) $f(x_1, x_2) = e^{-x_1 - x_2}$;
$0 < x_1 x_2 < \infty$

$f_1(x_1) = e^{-x_1}$; $0 < x_1 < \infty$
$f_2(x_2) = e^{-x_2}$; $0 < x_2 < \infty$
$\Rightarrow f(x_1 x_2) = f_1(x_1) f_2(x_2)$
$\Rightarrow x_1$ $\&$ x_2 ARE S.I.

$$E[e^{t(x_1+x_2)}] = \int_0^\infty \int_0^\infty e^{t(x_1+x_2)} e^{-x_1-x_2} dx_1 dx_2$$

$$= \int_0^\infty e^{tx_1} e^{-x_1} dx_1 \int_0^\infty e^{tx_2} e^{-x_2} dx_2$$

$$= \left[\int_0^\infty e^{tx} e^{-x} dx\right]^2$$

$$= \left[\int_0^\infty e^{-x(1-t)} dx\right]^2$$

$$= \left[\frac{1}{1-t}\right]^2 \quad ; 1-t>0$$
$$1 > t$$

$$= \frac{1}{(1-t)^2} \quad ; t < 1$$

(2-33)

$$f(x_i) = 3(1-x)^2 \quad ; \quad 0 < x_i < 1$$

$$i = 1, 2, 3, 4$$

$$Y = \min[X_1, X_2, X_3, X_4]$$

$$f(x_1, x_2, x_3, x_4) = 81(1-x_1)^2(1-x_2)^2(1-x_3)^2(1-x_4)^2 \quad ; 0<y<1$$

$$Pr[Y \geq y] = \left[\int_y^1 3(x-1)^2 dx\right]^4$$

$$= \left[(x-1)^3\Big|_y^1\right]^4$$

$$= \left[0 - (y-1)^3\right]^4$$

$$= (y-1)^{12} \quad ; \quad 0 < y < 1$$

$$F_Y(y) = Pr[Y \leq y] = 1 - Pr[Y \geq y] \quad ; 0 < y < 1$$

$$= 1 - (y-1)^{12}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= -12(y-1)^{11} \qquad 0 < y < 1$$

$$= 12(1-y)^{11}$$

(2-34) X_i = # OF SPOTS, $i = 1, 2, 3$

$p(X_i) = \frac{1}{6}$; $X_i = 1, 2, 3, 4, 5, 6$

$Y = \max.[X_i]$

$p[Y \le y] = Pr[X_1 \le Y, X_2 \le Y, X_3 \le Y]$

$= Pr[X_1 \le Y]\, Pr[X_2 \le Y]\, Pr[X_3 \le Y]$

$= Pr[X_1 \le Y]^3$

$Pr[X_1 = Y] = \frac{1}{6}$, $Y = 1, 2, 3, 4, 5, 6$

$Pr[X_1 \le Y] = \frac{Y}{6}$

$\Rightarrow Pr[Y \le Y] = \frac{Y^3}{6^3} = F_Y(Y)$

$f_Y(Y) = \frac{3Y^2}{6^3}$ $Y = 1, 2, 3, 4, 5, 6$

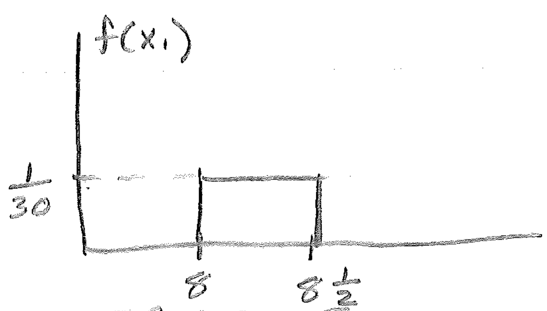
(2-35)

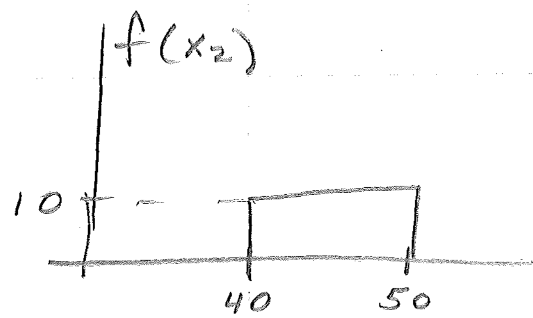
$f(x_1) = \frac{1}{30}$ $\quad 8{:}00 < X_1 \le 8{:}30$

$f(x_2) = \frac{1}{10}$ $\quad 40 < X_2 < 50$

FIND

$P_r[X_1 + X_2 \le 9{:}00]$





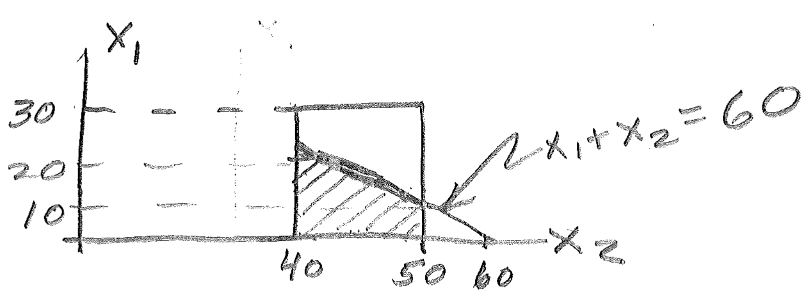
REDEFINE

$f(X_1) = \frac{1}{30}$ $\quad 0 < X_1 < 30$

$f(X_2) = \frac{1}{10}$ $\quad 40 < X_2 < 50]$

FIND

$P_r[X_1 + X_2 \le 60]$ $\qquad X_1$ & X_2 ARE S.I



$f(x_1, x_2) = \frac{1}{300}$ $\quad 0 < X_1 < 30, \; 40 < X_2 < 50$

$P_r[X_1 + X_2 \le 60]$

$$= \int_{X_1=0}^{20} \int_{X_2=40}^{60-X_1} \frac{1}{300}\, dx_2\, dx_1$$

$$= \frac{1}{300} \int_0^{20} [60 - X_1 - 40]\, dX_1 = \frac{1}{300} \int_0^{20} (20 - X_1)\, dX_1$$

$$= \frac{1}{300} \left[20X_1 - \frac{1}{2} X_1^2 \right]_0^{20}$$

$$= \frac{1}{300} [400 - 200] = 2/3$$

(2-36) $f(x_1 x_2 x_3) = \frac{1}{4}$ $(100)(010)(001)(111)$

$M(t_1 t_2 t_3) = E\left[e^{t_1 x_1 + t_2 x_2 + t_3 x_3}\right]$

$= \frac{1}{4}\left[e^{t_1} + e^{t_2} + e^{t_3} + e^{t_1+t_2+t_3}\right]$

$M(t_1, 0, 0) = \frac{1}{4}\left[2e^{t_1} + 2\right] = \frac{1}{2}\left[e^{t_1} + 1\right]$

$M(0, t_2 0) = \frac{1}{4}\left[2e^{t_2} + 2\right] = \frac{1}{2}\left[e^{t_2} + 1\right]$

$M(t_1 t_2 0) = \frac{1}{4}\left[e^{t_1} + e^{t_2} + 1 + e^{t_1 + t_2}\right]$

$= \frac{1}{4}\left[e^{t_1} + 1\right]\left[e^{t_2} + 1\right]$

$= M(t_1 00)\, M(0 t_2 0)$

$M(t_1 00) = \frac{1}{2}\left[e^{t_1} + 1\right]$

$M(00t_3] = \frac{1}{2}\left[e^{t_3} + 1\right]$

BY SYMMETRY:

$M(t_1 0 t_3) = M(t_1 00)\, M(00 t_3)$

$\&\ M(0, t_2 t_3) = M(0 t_2 0)\, M(00 t_3)$

NOW

$M(t_1 00)\, M(0 t_2 0)\, M(00 t_3)$

$= \frac{1}{8}\left[e^{t_1} + 1\right]\left[e^{t_2} + 1\right]\left[e^{t_3} + 1\right]$

$= \frac{1}{8}\left[1 + e^{t_1} + e^{t_2} + e^{t_1 + t_2}\right]\left[e^{t_3} + 1\right]$

$= \frac{1}{8}\left[e^{t_1} + e^{t_2} + 1 + e^{t_1 + t_2} + e^{t_3} + e^{t_1 + t_3} + e^{t_2 + t_3} + e^{t_1 + t_2 + t_3}\right]$

$\neq M(t_1 t_2 t_3)$

(2.37) LET R.V.'S $X_1 \ldots X_n$ HAVE A JOINT pdf $f(x_1 \ldots x_n)$. IF $X_1 \ldots X_n$ ARE MUTUALLY S.I IFF

$$f(x_1 \ldots x_2) = g_1(x_1) g_2(x_2) \cdots g_n(x_n)$$

$$\exists\, g_i(x_i) > 0 \quad \forall\, x_i \in A_i$$
$$i = 1, \ldots, n$$

(2-38) IF $X_1 \ldots X_n$ ARE MUTUALLY S.I., THEN

$$M(t_1 t_2 \ldots t_n) = M(t_1, 0, \ldots, 0) M(0, t_2 \ldots 0) \ldots M(0, 0 \ldots t_n)$$

11/3/76 BLD

(3-1) $M(t) = \left(\frac{1}{3} + \frac{2}{3}e^t\right)^5$

BINOMIAL DISTRIBUTION MGF:

$M(t) = [(1-p) + pe^t]^n$

$\Rightarrow n = 5, \; p = 2/3$

$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$

$= \binom{5}{x} \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{5-x}$

$Pr[x = 2, 3]$

$= \binom{5}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^3$

$+ \binom{5}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2$

$= \frac{5 \cdot 4}{2} \frac{4}{3^5} + \frac{5 \cdot 4}{2} \frac{8}{3^5}$

$= \frac{10}{3^5}[4 + 8]$

$3^5 = 81 \cdot 3 = 243$

$Pr[x = 2, 3] = \frac{120}{243} = \frac{40}{81}$

(3-2) $M(t) = \left(\frac{2}{3} + \frac{1}{3} e^t\right)^9$

BINOMIAL:

$M(t) = \left[(1-p) + pe^t\right]^n$

$\Rightarrow n = 9, \ p = \frac{1}{3}$

$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$

$= \binom{9}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}$

FOR BINOMIAL: $\mu = np = (9)\left(\frac{1}{3}\right) = 3$

$\sigma^2 = npq = 9\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = 2$

$\mu \pm 2\sigma = 3 \pm \sqrt{2} \approx 1.5, \ 5.5$

$\Pr\left[\mu - 2\sigma < X < \mu + 2\sigma\right]$

$= \Pr\left[1.5 < X < 5.5\right]$

$= \Pr\left[X = 2, 3, 4\right]$

$= \binom{9}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^7$

$+ \binom{9}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^6$

$+ \binom{9}{4} \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^5$

(3.3)

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

(a) $E\left(\frac{X}{n}\right) = \sum_x \binom{n}{x} p^x (1-p)^{n-x} X$

$$= \frac{1}{n} E[X] = \frac{1}{n} np = p$$

(b) $E\left[\left(\frac{X}{n} - p\right)^2\right] = E\left[\left(\frac{X-np}{n}\right)^2\right]$

$$= E\left[\left(\frac{X-np}{\sqrt{npq}}\right)^2\right]\frac{pq}{n}$$

BUT $\frac{X-np}{\sqrt{npq}} = \frac{X-\mu}{\sigma}$

$$\Rightarrow E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = 1$$

$$\Rightarrow E\left[\left(\frac{X}{n} - p\right)^2\right] = \frac{pq}{n} = \frac{p(1-p)}{n}$$

(3-4) $X_1 \; X_2 \; X_3$

$$f(x_i) = 3x_i^2 \quad ; \; 0 < x_i < 1 \quad i = 1,2,3$$

$$P(x_i) \geq \tfrac{1}{2}$$

$$= \int_{\frac{1}{2}}^{1} 3x_i^2 \, dx = x_i^3 \Big|_{\frac{1}{2}}^{1}$$

$$= 1 - \tfrac{1}{8} = \tfrac{7}{8}$$

$$P[2 \text{ ARE} \geq \tfrac{1}{2}]$$

$$= \binom{3}{2} \left(\tfrac{7}{8}\right)^2 \tfrac{1}{8}$$

$$= 3 \; \tfrac{49}{512} = \tfrac{147}{512}$$

$$\begin{array}{r} \overset{3}{6}4 \\ 8 \\ \hline 2 \end{array} \qquad \begin{array}{r} 49 \\ 3 \\ \hline 147 \end{array}$$

(3-5) $n = 3$ $\quad p = \frac{2}{3}$

FIND $P[Y \geq 2]$

$$f(x) = \binom{3}{x}\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^x$$

$$P[Y \geq 2] = P[X = 2] + P[X = 3]$$

$$= \binom{3}{2}\left(\frac{2}{3}\right)^2 \frac{1}{3} + \binom{3}{3}^1\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^0$$

$$= \frac{3 \cdot 4}{27} + \frac{8}{27} = 20/27$$

$n = 5$

$$f(Y) = \binom{5}{x}\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{5-x}$$

$$P_r[Y \geq 3]$$

$$= \binom{5}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2$$

$$+ \binom{5}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^1$$

$$+ \binom{5}{5}\left(\frac{2}{3}\right)^5$$

$$\frac{81/3}{243}$$

$$= \frac{5 \cdot \overset{2}{4}}{2 \cdot 1}\,\frac{8}{243} + 5\,\frac{16}{243} + \frac{32}{243}$$

$$= \frac{80 + 80 + 32}{243} = \frac{192}{243} = \frac{64}{81}$$

(3-6) $p = \frac{1}{4}$

$$Pr[Y \geq 1] = 1 - P(Y=0)$$
$$= 1 - \binom{n}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^n$$
$$= 1 - \left(\frac{3}{4}\right)^n \geq 0.7$$
$$-\left(\frac{3}{4}\right)^n \geq -0.3$$
$$\left(\frac{3}{4}\right)^n \leq 0.3$$
$$n \lg \frac{3}{4} \leq \lg 0.3$$

$n = 5$

(3-7)

$$f(x_1) = \binom{3}{x_1}\left(\frac{2}{3}\right)^{x_1}\left(\frac{1}{3}\right)^{3-x_1}$$

$$f(x_2) = \binom{4}{x_2}\left(\frac{1}{2}\right)^{x_1}\left(\frac{1}{2}\right)^{4-x_1}$$

$$= \binom{4}{x_2}\frac{1}{2}^4 = \frac{1}{16}\binom{4}{x_2}$$

$$P[X_1 = X_2] = P[X_1 = 0 \cap X_2 = 0]$$
$$+ P[X_1 = 1 \cap X_2 = 1]$$
$$+ P[X_1 = 2 \cap X_2 = 2]$$
$$+ P[X_1 = 3 \cap X_2 = 3]$$

$$= P[X_1 = 0]\,P[X_2 = 0]$$
$$+ P[X_1 = 1]\,P[X_2 = 1]$$
$$+ P[X_1 = 2]\,P[X_2 = 2]$$
$$+ P[X_1 = 3]\,P[X_2 = 3]$$

$$= \binom{3}{0}\left(\frac{1}{3}\right)^3\binom{4}{0}\frac{1}{16} + \binom{3}{1}\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2\frac{1}{16}\binom{4}{1}$$
$$+ \binom{3}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)\frac{1}{16}\binom{4}{2}$$
$$+ \binom{3}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{16}\right)\binom{4}{3}$$

$$= \frac{1}{27\cdot 16} + \frac{3\cdot 2\cdot 4}{27\cdot 16} + \frac{3\cdot 4\cdot \cancel{4}^{\,2}\cdot 3}{16\cdot 27\cdot \cancel{2}} \quad \frac{24}{2}$$
$$+ \frac{8\cdot 4}{27\cdot 16}$$

$$= \frac{1}{27\cdot 16}\left[1 + 24 + 72 + 32\right]$$

(3-8)

(a)
$$(p_4+p_1e^{t_1}+p_2e^{t_2}+p_3e^{t_3})^n$$
$$=\sum_{x_3=0}^{n}\binom{n}{x_3}(p_3e^{t_3})^{x_3}(p_4+p_1e^{t_1}+p_2e^{t_2})^{n-x_3}$$
$$=\sum_{x_3=0}^{n}\binom{n}{x_3}\sum_{x_2=0}^{n-x_3}(p_3e^{t_3})^{x_3}(p_2e^{t_2})^{x_2}\binom{n-x_3}{x_2}(p_4+p_1e^{t_1})^{n-x_3-x_2}$$
$$=\sum_{x_3=0}^{n}\sum_{x_2=0}^{n-x_3}\sum_{x_1=0}^{n-x_3-x_2}\binom{n}{x_3}\binom{n-x_3}{x_2}\binom{n-x_3-x_2}{x_1}(p_3e^{t_3})^{x_3}(p_2e^{t_2})^{x_2}(p_1e^{t_1})^{x_1}p_4^{\,n-x_3-x_2-x_1}$$

$$\binom{n}{x_3}\binom{n-x_3}{x_2}\binom{n-x_3-x_2}{x_1}$$
$$=\frac{n!}{x_3!\cancel{(n-x_3)!}}\;\frac{\cancel{(n-x_3)!}}{x_2!\cancel{(n-x_2-x_3)!}}\;\frac{\cancel{(n-x_3-x_2)!}}{x_1!\underbrace{(n-x_1-x_2-x_3)}_{x_4}!}$$
$$=\frac{n!}{x_1!\,x_2!\,x_3!\,x_4!}$$

(b) $0=t_2=t_3$
$$\Rightarrow\sum_{x_3=0}^{n}\sum_{x_2=0}^{n-x_3}\sum_{x_1=0}^{n-x_2-x_3}\frac{n!}{x_1!}$$
(HARD WAY)

$$(p_4+p_1e^{t_1}+p_2\cancel{e^{t_2}}+p_3\cancel{e^{t_3}})^n$$
BINOMIAL WITH $p=p_1$, $(1-p)=p_3+p_2+p_4$

(c) $t_3=0\Rightarrow(p_4+p_1e^{t_1}+p_2e^{t_2}+p_3)^n$

$$f(x_3|x_1,x_2)=\frac{f(x_1,x_2,x_3)}{f(x_1,x_2)}$$

$$E[x_3|x_1,x_2]=(n-x_1-x_2)\frac{p_3}{(1-p_1-p_2)}$$

(3-9) $X \sim b(2, p)$

$f(x) = \binom{2}{x} p^x (1-p)^{2-x}$

$P[X \geq 1] = 5/9 = 1 - P[X = 0]$

$= 1 - \binom{2}{0}(1-p)^2$

$4/9 = (1-p)^2$

$2/3 = 1 - p \Rightarrow p = \frac{1}{3}$

$Y \sim (4, p)$

$P[Y \geq 1] = 1 - P[Y = 0]$

$= 1 - \binom{4}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4$

$= 1 - \frac{16}{81}$

$= 65/81$

$\begin{array}{r} 81 \\ 16 \\ \hline 65 \end{array}$

(3-10) $x = r$ = MODE OF $b(n,p)$

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x+1) = \binom{n}{x+1} p^{x+1} (1-p)^{n-x-1}$$

$$\frac{f(x+1)}{f(x)} = \frac{\cancel{n!}\,(x+1)!\,(n-x-1)!}{x!\,(n-x)!\,\cancel{n!}} \; \frac{p}{1-p}$$

$$= \frac{x}{(n-x)} \; \frac{p}{1-p}$$

(3-11) $C_1 = \{x ; x = 1, 2, 3\}$

$p_1 = P[C_1] = 1/2$

$C_2 = \{x ; x = 4, 5\}$

$P(C_2) = 1/3 = p_2$

USE TRINOMIAL

$$f(x, y) = \frac{n!}{x!\, y!\, (n-x-y)!}\, p_1^{\,x} p_2^{\,y} p_3^{\,n-x-y}$$

$P(x_1 = 2, x_2 = 1)$ $\qquad \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

$$= \frac{5!}{2!\,1!\,2!} \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{3}\right)^1 \left(\tfrac{1}{6}\right)^2$$

$$= \frac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2}}{\cancel{2} \cdot \cancel{2}} \cdot \frac{1}{2 \cdot 2} \cdot \frac{1}{3} \cdot \frac{1}{2 \cdot 3 \cdot \cancel{2 \cdot 3}}$$

$$= \frac{5}{72}$$

(3-12) N.B.D:

$$f(Y) = \binom{Y+r-1}{r-1} p^r (1-p)^Y$$

$$= p^r [1-(1-p)]^{-r}$$

$$E[e^{tY}] = \Sigma_Y e^{tY} \binom{Y+r-1}{r-1} p^r (1-p)^Y$$

$$= \Sigma_Y \binom{Y+r-1}{r-1} p^r [e^t(1-p)]^Y$$

$$= p^r [1-e^t(1-p)]^{-r}$$

$$= \left[\frac{p}{1-e^t(1-p)}\right]^r = M(t)$$

$$\frac{d}{dt} M(t) = p^r (-r)(-e^t(1-p))[1-e^t(1-p)]^{-r-1}$$

$$= p^r r e^t (1-p) [1-e^t(1-p)]^{-r-1}$$

$$\frac{d}{dt} M(0) = p^r r (1-p) [\cancel{1-1}+p]^{-r-1}$$

$$= \frac{p^r r(1-p)}{p^{r+1}} = \frac{r(1-p)}{p} = \mu$$

$$\frac{d^2}{dt^2} M(t) = p^r r e^t (1-p) [1-e^t(1-p)]^{-r-1}$$

$$+ p^r r e^t (1-p) [+e^t(1-p)] (r+1)$$

$$[1-e^t(1-p)]^{-r-2}$$

$$E(x^2) = \frac{d^2}{dt^2} M(0)$$

$$= p^r r(1-p) [1-(1-p)]^{-r-1}$$

$$+ p^r r (1-p)^2 (r+1) [1-(1-p)]^{-r-2}$$

$$= \frac{p^r}{p^{r+1}} r(1-p) + \frac{p^r (1-p)^2 (r+1) r}{p^{r+2}}$$

$$= \frac{r(1-p)}{p} + \frac{r(r+1)(1-p)^2}{p^2}$$

$$\sigma^2 = E(x^2) - E(x)^2$$

$$= \frac{r(1-p)}{p} + \frac{r(r+1)(1-p)^2}{p^2} - \frac{r^2(1-p)^2}{p^2}$$

$$= \frac{r(1-p)}{p} + \frac{r(1-p)^2}{p^2}[r]$$

$$= \frac{r(1-p)}{p} + \frac{r^2(1-p)^2}{p^2}$$

(3-13) FOR TRINOMIAL:

$$M(t_1, t_2) = \left(p_1 e^{t_1} + p_2 e^{t_2} + (1 - p_1 - p_2)\right)^n$$

$$\frac{\delta M(t_1, t_2)}{\delta t_1} = n p_1 e^{t_1} \left(p_1 e^{t_1} + p_2 e^{t_2} + p_3\right)^{n-1}$$

$$\frac{\delta^2 M(t_1, t_2)}{\delta t_1 \, \delta t_2} = n(n-1) p_1 e^{t_1} p_2 e^{t_2} \left(p_1 e^{t_1} + p_2 e^{t_2} + p_3\right)^{n-2}$$

$$E[XY] = \delta^2 M(0,0) / \delta t_1 \, \delta t_2$$
$$= n(n-1) p_1 p_2 (1)^{n-2}$$
$$= n(n-1) p_1 p_2$$

$\mu_1 = n p_1 \qquad \mu_2 = n p_2$

$$\mathrm{cov}(X_1, X_2) = n(n-1) p_1 p_2 - n^2 p_1 \cancel{n} p_2$$
$$= n^2 p_1 p_2 - n p_1 p_2 - n^2 p_1 p_2$$
$$= -n p_1 p_2$$

(3-14)

$$f(x) = \binom{5}{x}\left(\frac{1}{2}\right)^5$$
$$= \binom{5}{x}\frac{1}{32}$$

P[AT LEAST 4]

$$= P[4] + P[5]$$
$$= \frac{1}{32}\left[\binom{5}{4} + \binom{5}{5}\right]$$
$$= 6/32 =$$

$$P[5 \text{ HEADS}] = \frac{1}{32}$$

P[5 HEADS / AT LEAST ~~5~~ 4 HEADS]

$$= \frac{P[5 \text{ HEADS} \cap \text{AT LEAST } 4 \text{ HEADS}]}{P[\text{AT LEAST } 4 \text{ HEADS}]}$$

5 HEADS $\in$ AT LEAST 4 HEADS

$$\Rightarrow P[5 \text{ HEADS} / \text{AT LEAST } 4] = \frac{P[5 \text{ HEADS}]}{P[\text{AT LEAST } 4]}$$

$$= \frac{1/32}{6/32} = \frac{1}{6}$$

(3-15) $n = 7$

$x_i = 1, 2, 3, 4, 5, 6$

$p(x_i) = \frac{1}{6}$

$1 - C_1 \quad 2 - C_2 \quad 3 - C_3$

$4 - C_4 \quad 5 - C_5 \quad 6 - C_6$

FIND

P[ALL SIDES APPEAR / 1 APPEARS TWICE]

$$= \frac{P[\text{ALL SIDES APEAR \& 1 APPEARS TWICE}]}{P[\text{1 APPEARS TWICE}]}$$

P[ALL & 1]

$$= \frac{7!}{(1!)^5\, 2!} \left(\frac{1}{6}\right)^7$$

$$= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{6^7}$$

$$P[\text{1 APPEARS TWICE}] = \frac{7!}{2!\,5!} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^5$$

$$\Rightarrow P[\text{ALL}/1] = \frac{\cancel{7!}}{\cancel{2}\cancel{6^7}} \cdot \frac{\cancel{2!}\, 5!\, \cancel{6^7}}{\cancel{7!}\, 5^5}$$

$$= \frac{5!}{5^5} = \frac{\cancel{5} \cdot 4 \cdot 3 \cdot 2}{\cancel{5} \cdot 5^4} = \frac{24}{625}$$

(3-16) a) $\mu'_k = E[X^k]$

$= \sum_{x=0}^{n} \binom{n}{x} x^k p^x (1-p)^{n-x}$

$\frac{d}{dp}\mu'_k = \sum \binom{n}{x} x^k \, x p^{x-1} (1-p)^{n-x}$

$\quad + \binom{n}{x} x^k p^x (x-n)(1-p)^{n-x-1}$

$= \sum \binom{n}{x} x^k p^x (1-p)^{n-x} \underbrace{\left[\frac{x}{p} + \frac{x-n}{1-p}\right]}_{\frac{x-xp+xp-np}{p(1-p)} = \frac{x-np}{p(1-p)}}$

$\Rightarrow \frac{d}{dp}\mu'_k = \sum \binom{n}{x} x^k p^x (1-p)^{n-x} \left(\frac{x-np}{p(1-p)}\right)$

$= \frac{1}{p(1-p)} \sum \binom{n}{k} x^{k+1} p^x (1-p)^{n-x}$

$- \frac{np}{p(1-p)} \sum \binom{n}{x} x^k (1-p)$

$= \frac{1}{p(1-p)} \mu'_{k+1} - \frac{np}{p(1-p)} \mu'_k$

$\Rightarrow \mu'_{k+1} = np\mu'_k + p(1-p)\frac{d}{dp}\mu'_k$

b) $\mu_k = E[(x-np)^k]$

$\cancel{= \sum_{x=0}^{n} \binom{n}{x} (x-np)^k p^x (1-p)^{n-x} \left[\frac{-kn}{x-np} + \frac{x}{p} + \frac{x-n}{(1-p)}\right]}$

$\cancel{\frac{d\mu_k}{dp} = \sum_{x=0}^{n} \binom{n}{x} (x-np)^k p^x (1-p)^{n-x}}$

$\cancel{\frac{-kn}{x-np} + \frac{x}{p} + \frac{x-n}{1-p} = \frac{-knp(1-p) + x(x-np)p}{(x-np)p(1-p)}}$

$\frac{d\mu_k}{dp} = \sum_{x=0}^{n} \binom{n}{x} (x-np)^{k-1} p^{x-1} (1-p)^{n-x-1}$

$\quad [(-nk)p(1-p) + x(x-np)(1-p)$

$\quad + (x-np)p(x-n)]$

$= \sum \binom{n}{x} (x-np)^{k-1} p^{x-1} (1-p)^{n-x-1}$

$\quad [-nkp + nkp^2 + x^2 - \cancel{x^2p} - 2np\,x$

$\quad + \cancel{x^2}p - \cancel{xpn} - np^2x + n^2p^2]$

(3-17) $f(x_1, x_2) = \binom{x_1}{x_2}\left(\frac{1}{2}\right)^{x_1}\left(\frac{x_1}{15}\right)$

$x_2 = 0, 1, 2, \ldots x_1$

$x_1 = 1, 2, 3, 4, 5$

$$E(x_2) = \sum_{x_1=1}^{5} \frac{x_1}{15} \sum_{x_2=0}^{x_1} x_2 \underbrace{\left(\frac{1}{2}\right)^{x_1}\binom{x_1}{x_2}}_{\text{OF THE FORM } b(x_1, \frac{1}{2})}$$

$$\Rightarrow E[x_2] = \sum_{x=1}^{5} \frac{x_1}{15} \underbrace{\left(\frac{x_1}{2}\right)}_{np}$$

$$= \frac{1}{30} \sum_{x=1}^{5} x_1^2$$

$$= \frac{1}{30}[1 + 4 + 9 + 16 + 25] = \frac{55}{30} = \frac{11}{6}$$

b. FIND $E[X_2 | x_1]$

FIND $f(x_2|x_1) = \frac{f(x_1 x_2)}{f(x_1)}$

$$f(x_1) = \sum_{x_2=0}^{x_1} \binom{x_1}{x_2}\left(\frac{1}{2}\right)^{x_1}\left(\frac{x_1}{15}\right)$$

$$= \frac{x_1}{15} \sum \binom{x_1}{x_2}\left(\frac{1}{2}\right)^{x_1}$$

$$= \frac{x_1}{15}(1) = x_1/15$$

$$\Rightarrow f(x_2|x_1) = \binom{x_1}{x_2}\left(\frac{1}{2}\right)^{x_1}$$

$$E[X_2|X_1] = \sum_{x_2=0}^{x_1} \binom{x_1}{x_2} X_2 \left(\frac{1}{2}\right)^{x_1}$$

$$= x_1/2$$

c. $E[E(x_2/x_1)] = E\left[\frac{x_1}{2}\right]$

$$= \sum_{x_1=1}^{5} \frac{x_1^2}{30} \sum_{x=0}^{x_1} \left(\frac{1}{2}\right)^{x}\binom{x_1}{x_2}$$

$$= \sum_{x=1}^{5} x_1^2/30 = 11/6$$

SAME AS a

(3-18) $C_1 = HHH$ $P(C_1) = \frac{1}{8}$

$C_2 = HHT$ $P(C_2) = \binom{3}{2}\frac{1}{8} = \frac{3}{8}$

$C_3 = TTH$ $P(C_3) = 3/8$

$C_4 = TTT$ $P(C_4) = 1/8$

$n = 10$

$X_i = C_1$ OR C_4 $\Rightarrow P[X] = 2/8$

$Y_i = C_2$ OR C_3 $\Rightarrow P[Y] = 6/8$

$X = 10 - Y$

$$P(X) = \binom{10}{X}\left(\frac{2}{8}\right)^X\left(\frac{6}{8}\right)^{10-X}$$

$$E[P_X(X)] = np = 10\left(\frac{2}{8}\right) = \frac{20}{8} = \frac{10}{4} = \frac{5}{2}$$

$$E[P_Y(Y)] = 10\,\frac{6}{8} = \frac{60}{8} = \frac{30}{4} = \frac{15}{2}$$

$$\begin{array}{r} 15 \\ 9 \\ \hline 135 \end{array}$$

$$E[6XY] = 6\,E[X]\,E[Y]$$

$$\cancel{= 36\left[\frac{15 \cdot 63}{4}\right] = 135}$$

$$= 6 \cdot \frac{15 \cdot 5}{2 \cdot 2} = \frac{450}{4} = \cancel{\frac{225}{2}}$$

$$= \frac{112}{4}$$

(3-19) $P_r(x=1) = P(x=2)$

$$P(x) = \frac{\mu^x e^{-\mu}}{x!}$$

$$P(1) = \frac{\mu \cancel{e^{-\mu}}}{1!} = \frac{\mu^2 \cancel{e^{-\mu}}}{2!} = P(2)$$

$$2\mu = \mu^2 \Rightarrow 2 = \mu$$

$$P(x) = \frac{2^x e^{-2}}{x!}$$

$$P(x=4) = \frac{2^4 e^{-2}}{4!} = 0.09$$

(3-20) $M(t) = e^{4(e^t - 1)}$

$\Rightarrow m = 4 = \mu$

$\sigma^2 = m = 4 \Rightarrow \sigma = 2$

$\mu \pm 2\sigma = 4 \pm 4 = (0, 8)$

$P_r[0 \leq x < 8] = P[x \leq 7] - P[x \leq 0]$

$= \cancel{P[x < 8]} = \cancel{P[x \leq 8]}$

FOR $\mu = 4$, FROM TABLE ON p. 398

$\cancel{P[k = 7] = 0.999} \quad \cancel{P[x = 0] = .015}$

$P[x \leq 7] = 0.949 \qquad P[x = 0] = 0.018$

$\Rightarrow P_r[0 < x < 8] = 0.931$

(3-21) 13.5% $\Rightarrow$ NO TYPING ERRORS

$$f(x) = \frac{\mu^x e^{-\mu}}{x!}$$

WHAT IS μ?

$\mu = 1 - 0.135 = .865$

= PROPORTION OF PAGES WITH MISTAKES

$$f(1) = (.865)^1 e^{-0.865}$$

$= 0.364$

(3-22)

$$f(x) = \frac{4}{x}\, f(x-1)$$

TRY POISSON:

$$f(x) = \mu^{+x} e^{-\mu} / x!$$

$$f(x-1) = \mu^{x+1} e^{-\mu} / (x-1)!$$

$$\frac{f(x)}{f(x-1)} = \frac{\mu^{x}}{\mu^{x+1}} \frac{(x-1)!}{x!} = \frac{1}{x\mu}$$

$\Rightarrow$ POISSON R.V. WITH $\mu = \frac{1}{4}$

(3-23) $\mu = 100$

FIND, USING CHEBYCHEF:

$$Pr[75 < X < 125]$$
$$= Pr[-25 < X - \mu < 25]$$
$$= Pr[|X - \mu| < 25] = 1 - P[|X - \mu| \geq 25]$$

CHEBY SAYS

$$Pr[|X - \mu| \geq k\sigma] \leq \frac{1}{K^2}$$
$$\sigma = \sqrt{100} = 10 \Rightarrow 25 = k\,10 \Rightarrow k = 2.5$$

THUS $1/K^2 = 0.16$

$$\Rightarrow$$

$$Pr[|X - \mu| \geq k\sigma] \leq 0.16$$
$$1 - Pr[|X - \mu| \geq 25] \geq 1 - 0.16 = 0.84$$

(3-24) $\frac{\delta}{\delta w} g(x,w) = -\lambda g(x,w) + \lambda g(x-1;w)$

FOR $x=1$

$$g(1,w) = \lambda w e^{-\lambda w}$$

$$\frac{\delta}{\delta w} g(1,w) = \lambda e^{-\lambda w} - \lambda(\lambda w e^{\lambda w})$$

$$= \lambda g(0,w) - \lambda g(1,w)$$

ASSUME FOR $x-1$:

$$\frac{\delta}{\delta w} g(x-1,w) = -\lambda g(x-1,w) + \lambda g(x-2;w)$$

THEN, FOR x

$$g(x,w) = \frac{(\lambda w)^x e^{-\lambda w}}{x!}$$

$$\frac{\delta g(w,x)}{\delta w} = \lambda x \frac{(\lambda w)^{x-1} e^{-\lambda w}}{x!}$$

$$- \lambda \frac{(\lambda w)^x e^{-\lambda w}}{x!}$$

$$= \frac{\lambda(\lambda w)^{x-1} e^{-\lambda w}}{(x-1)!} - \lambda \frac{(\lambda w)^x e^{-\lambda w}}{x!}$$

$$= \lambda g(x-1,w) - \lambda g(x,w)$$

QED

(3-25) $X = \#$ CHOCO DROPS $\sim P(\mu)$

$$P(\mu) = \frac{\mu^x e^{-\mu}}{x!}$$

FIND $\mu \ni$

$$P(\mu \geq 2) > 0.99$$

$$P[\mu \geq 2] = 1 - P[X \leq 1] \geq 0.99$$

$$-P[X \leq 1] \geq -0.01$$

$$P[X \leq 1] \leq 0.01$$

FROM TABLE ON Pg 398, WE GOTTA HAVE AT LEAST $\mu = 9$ DROPS

(3-26) $\mu_k' = E[X^k]$

SHOW

$$\mu_{k+1}' = \mu\left[\mu_k' + \frac{d\mu_k'}{d\mu}\right]$$

$$\mu_k' = E[X^k] = \sum_{x=0}^{\infty} \frac{x^k \mu^x e^{-\mu}}{x!}$$

$$\frac{d\mu_k'}{d\mu} = \sum_{x=0}^{\infty} \frac{[x^k x \mu^{x-1} - x^k \mu^x] e^{-\mu}}{x!}$$

$$= \sum \frac{x^{k+1} \mu^x}{\mu\, x!} - \sum \frac{x^k \mu^x e^{-\mu}}{x!}$$

$$= \frac{1}{\mu} \mu_{k+1}' - \mu_k'$$

$$\Rightarrow \mu_{k+1}' = \mu\left[\mu_k' + \frac{\delta \mu_k'}{\delta \mu}\right]$$

(3-27) $f(x,Y) = \dfrac{e^{-2}}{x!\,(Y-x)!}$

$Y = 0, 1, 2, \ldots \infty$

$x = 0, 1, 2, \ldots, Y \qquad Y \geq x \qquad x \leq Y$

(a) $M(t) = E\left[e^{t_1 x + sY} \right]$

$$= \sum_{Y=0}^{\infty} e^{sY} \sum_{x=0}^{Y} \frac{e^{tx} e^{-2}}{x!\,(Y-x)!}$$

$$= e^{-2} \sum_{Y=0}^{\infty} \frac{e^{sY}}{Y!} \sum_{x=0}^{Y} \binom{Y}{x} e^{tx}$$

$$= e^{-2} \sum_{Y=0}^{\infty} \frac{e^{sY}}{Y!} \sum_{x=0}^{Y} \binom{Y}{x} (e^{t})^{x} (1)^{Y-x}$$

$$e^{-2} \sum_{Y=0}^{\infty} \frac{e^{sY}}{Y!} (1 + e^{t})^{Y}$$

$$= e^{-2} \sum_{Y=0}^{\infty} \frac{\left[e^{s}(1+e^{t})\right]^{Y}}{Y!}$$

$$= e^{e^{s}(1+e^{t}) - 2}$$

(b) $\frac{d}{dt} e^{e^{s}(1+e^{t})-2} = \frac{d}{dt} e^{e^{s}} e^{e^{s+t}} e^{-2}$

$$= e^{s+t} e^{e^{s+t}} e^{e^{s}} e^{-2}$$

$\frac{d}{dt} M(0,0) = E[x] = 1 \times e^{2} e^{-2} = 1 = \mu_x$

$\frac{d}{ds} M(0,0) = E[Y] = e^{s} e^{e^{s}} e^{e^{s+t}} e^{-2}$

$\quad + e^{e^{s}} e^{s+t} e^{e^{s+t}} e^{-2}$

$= 1 \cdot e^{2} e^{-2} + e^{2} e^{-2} = 2 = \mu_Y$

$\frac{d^2}{dt^2} M(0,0) = E[x^2]$

$\Rightarrow e^{s+t} e^{s+t} e^{e^{s+t}} e^{e^{s}} e^{-2}$

$\quad + e^{s+t} e^{e^{s+t}} e^{e^{s}} e^{-2} \Rightarrow e^{2} e^{-2} + e^{2} e^{-2} = 2$

$\Rightarrow \sigma_x^{2} = 2 - 1 = 1$

$\frac{d^2}{ds^2} M(0,0) = E(Y^2)$

$\Rightarrow \left[1 + e^{s} + e^{s+t}\right] e^{s} e^{e^{s}} e^{e^{s+t}} e^{-2}$

$\quad + \left[e^{s} + e^{s+t} + e\right] e^{e^{s}} e^{s+t} e^{e^{s+t}} e^{-2}$

$\Rightarrow 3 + 3 = 6 \quad \Rightarrow \sigma_Y^{2} = 6 - 4 = 2$

$$\frac{\delta^2}{\delta s\,\delta t} M(0,0) = E[X,Y]$$

$$\Rightarrow [1 + e^{s+t} + e^{s}] e^{s+t} e^{e^{s+t}} e^{e^{s}} e^{-2}$$

$$\Rightarrow 3$$

$$cov = E[XY] - \mu_x \mu_Y$$

$$= 3 - (1)(2) = 1$$

$$\rho = \frac{cov}{\sigma_1 \sigma_2} = \frac{1}{1\sqrt{2}} = \frac{2}{\sqrt{2}}$$

(c) $E[X|Y]$

$$f(x|Y) = f(x,Y) / f(Y)$$

$$f(Y) = \sum_x f(x,Y)$$

$$= \sum_0^Y \frac{e^{-2}}{x!(Y-x)!}$$

$$= \frac{1}{Y!} e^{-2} 2^Y \underbrace{\sum_0^Y \binom{Y}{x} \left(\frac{1}{2}\right)^Y}_{b(Y,\frac{1}{2})}$$

$$= \frac{1}{Y!} e^{-2} 2^Y$$

$$\Rightarrow f(x/Y) = \frac{\cancel{e^{-2}}}{x!(Y-x)!} \Big/ \frac{1}{Y!} \cancel{e^{-2}} 2^Y$$

$$= \binom{Y}{x} \left(\frac{1}{2}\right)^Y$$

$$E[(x|Y)] = \sum_{x=0}^{Y} x \underbrace{\binom{Y}{x}\left(\frac{1}{2}\right)^Y}_{b(\frac{1}{2},Y)}$$

$$= Y/2$$

(3-25) $M(t) = (1-2t)^{-6}$

χ^2 WITH $\frac{r}{2} = 6 \Rightarrow r = 12$ DOF

$P_r[X < 5.23] = 0.050$

(from TABLE ON $P_{\underline{5}}$ 399)

(3-29) $X \sim \chi^2(5)$

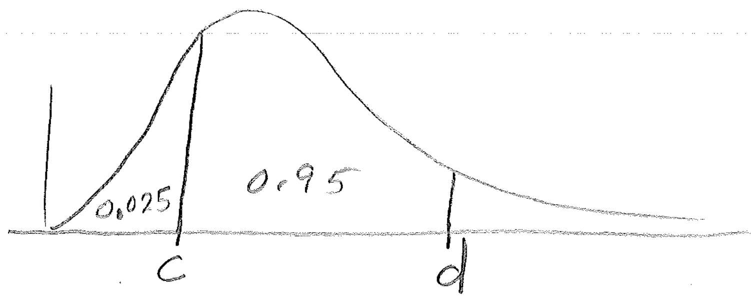
$Pr[X<c] = 0.025$

From 399 Table, $c = 0.831$

$Pr[c<x<d] = Pr[x<d] - Pr[x<c]$

$= Pr[x<d] - 0.025 = 0.95$

$\Rightarrow Pr[x<d] = 0.975$



$\Rightarrow d = 12.8$ (FROM 399 TABLE)

(3-30) $\alpha = 3, \ \beta = 4$

$$P_r[3.28 < X < 25.2]$$

$$= \int_{3.28}^{25.2} \frac{1}{\Gamma(3)\,4^3} X^2 e^{-X/4}\,dx$$

LET $Y = X/2$

$$d\cancel{x} = \tfrac{1}{2}\,dY$$

$$= \int_{1.64}^{12.6} \frac{1}{\Gamma(\underset{\alpha}{3})\,\underset{\beta}{4}^3} (2Y)^{\overset{\alpha-1}{2}} e^{-Y/\underset{\beta}{2}}\,dx$$

$$= \frac{4}{2^3} \int_{1.64}^{12.6} \underbrace{\frac{1}{\Gamma(\alpha=3)\,2^3}\, Y^2 e^{-Y/2}\,dx}_{\chi^2(r=2\alpha=6)}$$

$$= \frac{1}{2}\left[P_r[X<12.6] - P_r[X<1.64]\right]$$

$$= \frac{1}{2}[0.95 - 0.5]$$

$$= \frac{1}{2}\,0.45 = 0.225$$

(3-31) $E[X^m] = 2^m (m+1)!$, $m = 0, 1, \ldots$

$$M(t) = \sum_{m=0}^{\infty} \frac{E[X^m]\, t^m}{m!}$$

$$= \sum_{m=0}^{\infty} \frac{2^m (m+1)!\, t^m}{m!}$$

$$= \sum_{m=0}^{\infty} (m+1)(2t)^m$$

$$= \underbrace{\sum_{m=0}^{\infty} m(2t)^m}_{S_1} + \sum_{m=0}^{\infty} (2t)^m$$

$$S_1 = \sum_{m=0}^{\infty} m(2t)^m$$

$$= 2t \sum_{m=0}^{\infty} m(2t)^{m-1}$$

$$\frac{1}{2t} S_1 = \sum_{m=0}^{\infty} m(2t)^{m-1}$$

$$\int^t \frac{1}{2t} S_1 = \sum_{m=0}^{\infty} \int^t m(2t)^{m-1}$$

$$\sum_{m=0}^{\infty} \frac{1}{2}(2t)^m$$

$$\Rightarrow \int^t \frac{1}{t} S_1 = \sum_{m=0}^{\infty} (2t)^m$$

$$= \frac{1}{1-2t}$$

$$\Rightarrow \frac{S_1}{t} = \frac{d}{dt} \frac{1}{1-2t} = \frac{d}{dt}(1-2t)^{-1}$$

$$= (-1)(-2)(1-2t)^{-2}$$

$$= \frac{2}{(1-2t)^2}$$

$$\Rightarrow S_1 = 2t/(1-2t)^2$$

$$\therefore M(t) = \frac{2t}{(1-2t)^2} + \frac{1}{1-2t}$$

$$= \frac{2t + (1-2t)}{(1-2t)^2} = \frac{1}{(1-2t)^2}$$

$\Rightarrow$ GAMMA WITH $\beta = \alpha = 2$

$\Rightarrow \chi^2$, $r = 2\alpha = 4$

OR

$\chi^2(4)$

(3.32)

$$I_K = \int_0^\infty \frac{1}{\Gamma(K)} z^{K-1} e^{-z} dz$$
$$= \frac{1}{(K-1)!} \int_0^\infty z^{K-1} e^{-z} dz$$

PARTS INTEGRATION:

$$U = \frac{1}{(K-1)!} z^{K-1} \qquad dV = e^{-z} dz$$
$$dU = \frac{1}{(K-2)!} z^{K-2} \qquad V = -e^{-z}$$

$$\Rightarrow I_K = \left.\frac{-z^{K-1} e^{-z}}{(K-1)!}\right|_U^\infty + \int_U^\infty \frac{1}{(K-2)!} z^{K-2} e^{-z} dz$$

$$= \frac{U^{K-1} e^{-z}}{(K-1)!} + \int_U^\infty \frac{1}{(K-2)!} z^{K-2} e^{-z} dz$$

$$= \frac{U^{K-1} e^{-z}}{(K-1)!} + I_{K-1}$$

WE HAVE SET UP A DIFFERENCE RECURSIVE RELATION:

$$I_K = \frac{U^{K-1} e^{-z}}{(K-1)!} + \frac{U^{K-2} e^{z}}{(K-2)!} + I_{K-2}$$

OR, EXTENDING FURTHER:

$$I_K = \frac{U^{K-1} e^{-z}}{(K-1)!} + \frac{U^{K-2} e^{z}}{(K-2)!} + \frac{U^{K-3} e^{z}}{(K-3)!}$$

$$+ \ldots + \underbrace{\frac{U^{K-K} e^{-z}}{(K-K)!}}_{= \frac{U^0 e^{-z}}{0!}}$$

$$\Rightarrow I_K = \int_U^\infty \frac{1}{\Gamma(K)} z^{K-1} e^{-z} dz$$

$$= \sum_{x=0}^{K-1} \frac{U^x e^{-z}}{x!}$$

$X_1, X_2, X_3 \sim e^{-x_i} \quad i = 1, 2, 3$

(3-33) $Y = Min(X_1, X_2, X_3)$

$P[Y \geq y] = P[X_1 \geq y, X_2 \geq y, X_3 \geq y]$

$= P[X_1 \geq y] \, P[X_2 \geq y] \, P[X_3 \geq y]$

$= P[X_1 \geq y]^3$

$= \left[\int_y^\infty e^{-x} dx\right]^3$

$= \left[-e^{-x} \Big|_y^\infty\right]^3 = e^{-3y} = 1 - F(y)$

$\Rightarrow F(y) = 1 - e^{-3y}$

$f(y) = 3e^{-3y} \; ; \; 0 < y < \infty$

(3-34) $f(x) = \frac{1}{\beta^2} x e^{-x/\beta}$

MODE = 2

$\Rightarrow \frac{d}{dx} f(2) = 0$

$\frac{d}{dx} f(x) = \frac{1}{\beta^2}\left[1 - \frac{x}{\beta}\right] e^{-x/\beta}$

$\Rightarrow \beta = 2$

$\beta = 2 \Rightarrow \chi^2(2\alpha)$ DISTRIBUTION

$\alpha = 2$

$\Longrightarrow \chi^2(4)$ DISTRIBUTION

$P_r[X < 9.49] = 0.95$

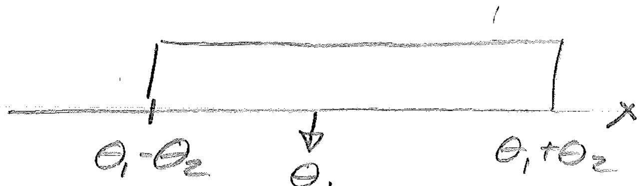
(FROM TABLE ON p 339)

(3-35) $X \sim \chi^2(8)$

$E[X] = r = 8$

$\sigma^2 = 2r = 16 \Rightarrow E[X^2] = 16 + 64 = 80$

$f(x) = \frac{1}{2\theta_2} \qquad \theta_1 - \theta_2 < x < \theta_1 + \theta_2,\ 0 < \theta_2$



$E[X] = \frac{1}{2\theta_2}\int_{\theta_1-\theta_2}^{\theta_1+\theta_2} x\,dx$

$= \frac{1}{4\theta_2}\left[(\theta_1+\theta_2)^2 - (\theta_1-\theta_2)^2\right]$

$= \frac{1}{4\theta_2}\left[\cancel{\theta_1^2} + 2\theta_1\theta_2 + \cancel{\theta_2^2} - \cancel{\theta_1^2} + 2\theta_1\theta_2 - \cancel{\theta_2^2}\right]$

$= \frac{4\theta_1\cancel{\theta_2}}{\cancel{4\theta_2}} = \theta_1$ (OBVIOESLY!)

$\Rightarrow \theta_1 = 8$

$E(X^2) = \frac{1}{2\theta_2}\int_{\theta_1-\theta_2}^{\theta_1+\theta_2} x^2\,dx$

$= \frac{1}{6\theta_2}\left[(\theta_1+\theta_2)^3 - (\theta_1-\theta_2)^3\right]$

$= \frac{1}{6\theta_2}\Big[(\cancel{\theta_1^3} + 3\theta_1^2\theta_2 + \cancel{3\theta_1\theta_2^2} + \theta_2^3)$

$\qquad - (\cancel{\theta_1^3} - 3\theta_1^2\theta_2 + \cancel{3\theta_1\theta_2^2} - \theta_2^3)\Big]$

$= \frac{1}{6\theta_2}\left[6\theta_1^2\theta_2 + 2\theta_2^2\right]$

$= \frac{1}{6\theta_2}\left[6 \cdot 8^2\theta_2 + 2\theta_2^2\right]$

$\sigma^2 = \frac{1}{6\theta_2}\left[6\cdot 64\cdot\theta_2 + 2\theta_2^2\right] - 64 = 16$

$\Rightarrow \frac{\cancel{6}\cdot 64\cancel{\theta_2}}{\cancel{6}\cancel{\theta_2}} + \frac{\cancel{2}\theta_2^{\cancel{2}}}{3\cancel{6}\cancel{\theta_2}} = 80$

$64 + \theta_2/3 = 80$

$\theta_2/3 = 16$

$\theta_2 = \sqrt{48} = 4\sqrt{3}$

$\neq 4\sqrt{3}$

(3-36) $X \sim POIS(m)$

$m \sim GAMMA(\alpha = 2, \beta = 1)$

$$f(x|m) = \frac{m^x e^{-m}}{x!}$$

$$f(m) = \Gamma(2,1) = m e^{-m}$$

$$f(x,m) = f(x|m) f(m)$$

$$= \frac{m^x e^{-m}}{x!} m e^{-m} = \frac{m^{x+1} e^{-2m}}{x!}$$

$$f(x) = \int_0^\infty f(x,m)\, dm$$

$$= \frac{1}{x!} \int_0^\infty m^{x+1} e^{-2m}\, dm$$

$$m' = 2m \Rightarrow dm = \tfrac{1}{2} dm'$$

$$f(x) = \frac{1}{x!} \frac{1}{2} \int_0^\infty \left(\frac{m'}{2}\right)^{x+1} e^{-m'}\, dm'$$

$$= \frac{1}{x!} \frac{1}{2} \frac{1}{2^{x+1}} \int_0^\infty m'^{(x+2)-1} e^{-m'} dm'$$

$$= \frac{1}{x!} \frac{1}{2^{x+2}} \Gamma(x+2)$$

$$= \frac{(x+1)!}{x!\, 2^{x+2}} = \frac{x+1}{2^{x+2}}$$

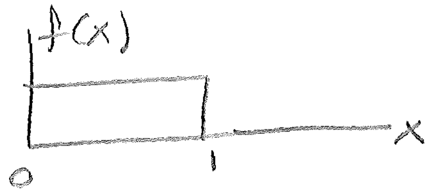
$$f(0) = \frac{1}{4}$$

$$f(1) = \frac{2}{8} = \frac{1}{4}$$

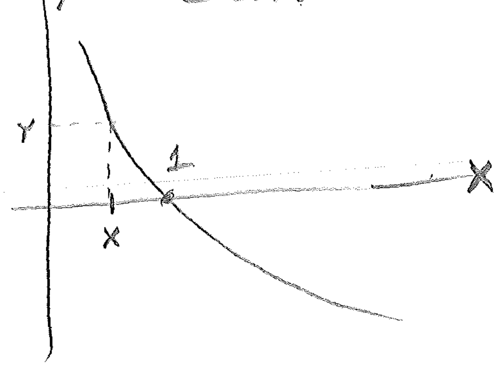
$$f(2) = \frac{3}{2^4} = \frac{3}{16}$$

$$P[X = 0, 1, 2] = \frac{1}{2} + \frac{3}{16} = \frac{11}{16}$$

(3-37)



$$Y = -2\ln X \Rightarrow X = e^{-Y/2}$$



$$P[Y \le y] = [-2\ln X \le y]$$
$$= P[\ln X \ge y/-2]$$
$$= P[X \ge e^{-y/2}]$$
$$= \int_{e^{-y/2}}^{1} dx = 1 - e^{-y/2}$$

THUS

$$F(y) = \begin{cases} 0 & ; \; y \le 0 \\ 1 - e^{-y/2} & ; \; y > 0 \end{cases}$$

$$f(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{2} e^{-y/2} & ; \; y > 0 \end{cases}$$

NOTE:

$$\chi^2(2) \Rightarrow \frac{1}{\Gamma(2/2)\, 2^{2/2}} x^{\frac{2}{2}-1} e^{-x/2}$$
$$= \frac{1}{2} e^{-x/2}$$

$$\Rightarrow Y \sim \chi^2(2)$$

(3-38)

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\,dw$$

$$N(-x) = \int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\,dw$$

$$= \frac{1}{2} - \int_{-x}^{0} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\,dw$$

$w' = -w$ GIVES

$$N(-x) = \frac{1}{2} - \int_{0}^{x} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\,dw$$

BUT

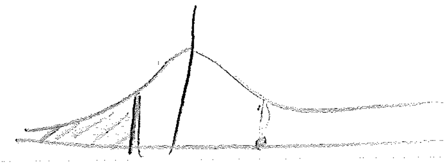
$$N(x) = \frac{1}{2} + \int_{0}^{x} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\,dw$$

$$\Rightarrow N(-x) = 1 - N(x)$$

(3-39) $X \sim n(75, 100)$

(a) $P_r[X<60] = P\left[\frac{x-75}{10} \le \frac{60-75}{10} = -1.5\right]$

$P_r\left[\frac{x-\mu}{\sigma} \le -1.5\right]$



$= 1 - P_r\left[\frac{x-\mu}{\sigma} \le 1.5\right]$

$= 1 - 0.933 = 0.067$

(b) $P_r[70 < x < 100]$

$= P_r[X<100] - P[x \le 70]$

$P_r[x<100] = P\left[\frac{x-75}{10} < 2.5\right] = 0.994$

$P_r[X<70] = P\left[\frac{x-75}{10} < -.5\right]$

$= 1 - P\left[\frac{x-75}{10} < .5\right]$

$\cancel{= 1 - 0.999 = 0.001}$

$\cancel{\Rightarrow P_r[70<x<100]}$

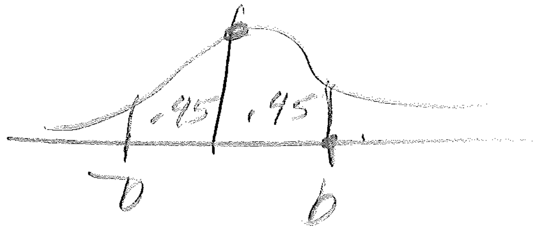
$= 1 - 0.691 = .309$

$P_r[70 < x < 100] = .994 - .309$

$= .685$

(3-40) $X \sim (\mu, \sigma^2)$

$Pr\left[-b < \frac{X-\mu}{\sigma} < b\right] = 0.90$



$Pr\left[\frac{X-\mu}{\sigma} < b\right] = 0.95$

FROM 400 TABLE

$b = 1.645$

(3-41)

$$Pr[X<89] = P\left[\frac{x-\mu}{\sigma} \le \frac{89-\mu}{\sigma}\right] = 0.9$$

$$\Rightarrow \frac{89-\mu}{\sigma} = 1.282$$

$$Pr[X<94] = P\left[\frac{x-\mu}{\sigma} \le \frac{94-\mu}{\sigma}\right] = 0.95$$

$$\frac{94-\mu}{\sigma} = 1.645$$

$$\Rightarrow \frac{94-\mu}{89-\mu} = \frac{1.645}{1.282}$$

$$94-\mu = \frac{1.645}{1.282}(89-\mu)$$

$$-\mu + \frac{1.645}{1.282}\mu = \frac{89 \cdot 1.645}{1.282} - 94$$

$$\mu\left(\frac{1.645}{1.282} - 1\right) = \frac{89 \cdot 1.645}{1.282} - 94$$

$$\mu(.28315) = 20.200$$

$$\mu = 71.34$$

$$\frac{94-\mu}{\sigma} = 1.645 \Rightarrow \sigma = \frac{94-\mu}{1.645}$$

$$= 13.77$$

$$\sigma^2 = 190$$

(3-42)

$$f(x) = Ce^{-x^2+4x}$$
$$= Ce^{-(x^2-4x)}$$
$$= Ce^{-[(x-2)^2-4]}$$

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = Ce^{+4}\int_{-\infty}^{\infty} e^{-(x-2)^2}dx$$

$$= Ce^{+4}\int_{-\infty}^{\infty} e^{-x^2}dx$$

$$x' = \frac{x}{\sqrt{2}} \Rightarrow dx = \frac{dx'}{\sqrt{2}}$$

$$1 = \frac{Ce^{+4}}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-\frac{x'^2}{2}}dx'$$

$$= \frac{Ce^{+4}}{\sqrt{2}}\sqrt{2\pi} = C\sqrt{\pi}\,e^{+4}$$

$$\Rightarrow C = \frac{e^{-4}}{\sqrt{\pi}}$$

(3-43)

$$n = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\frac{dn}{dx} = \frac{-1}{\sqrt{2\pi}\,\sigma}\left[\frac{d}{dx}\frac{(x-\mu)^2}{2\sigma^2}\right] e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{-1}{\sqrt{2\pi}\,\sigma^3}(x-\mu)\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\frac{d^2n}{dx^2} = \frac{-1}{\sqrt{2\pi}\,\sigma^3}\left[1 - (x-\mu)\frac{d}{dx}\frac{(x-\mu)^2}{2\sigma^2}\right] e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= 0$$

$$\Rightarrow 1 = (x-\mu)\frac{(x-\mu)}{\sigma^2} = \frac{(x-\mu)^2}{\sigma^2}$$

$$(x-\mu)^2 = \sigma^2$$

$$x - \mu = \pm\sigma$$

$$x = \mu \pm \sigma \Leftarrow \text{INFLEC PTS.}$$

(3-44)

$$P[X \le p] = 0.9$$

$$P\left[\frac{X-\mu}{\sigma} \le \frac{p-65}{5}\right] = 0.9$$

$$\frac{p-65}{5} = 1.282$$

$$p = 5 \times 1.282 + 65 = 71.41.$$

(3-45)

$M(t) = e^{3t+8t^2}$

$\mu = 3 \qquad \frac{\sigma^2}{2} = 8 \Rightarrow \sigma^2 = 4$

$Pr[-1 < X < 9]$

$= Pr\left[\frac{-1-3}{2} < Z < \frac{9-3}{2}\right]$

$= Pr[-2 < Z < 3]$

$= Pr[Z < 3] - Pr[Z < -2]$

$Pr[Z < 3] = 0.999$

$Pr[Z < -2] = 1 - P[Z < 2]$

$= 1 - 0.977$

$= .023$

$\Rightarrow Pr[-1 < X < 9] = 0.999$

$\quad - .023$

$= 0.976$

(3.46) $f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}$; $0 < x < \infty$

$$E[x] = \frac{2}{\sqrt{2\pi}} \int_0^\infty x e^{-x^2/2} dx$$

$$= \frac{2}{\sqrt{2\pi}} \left[-e^{-x^2/2} \right]_0^\infty$$

$$= 2/\sqrt{2\pi} = \sqrt{2/\pi}$$

$$E(x^2) = \frac{2}{\sqrt{2\pi}} \int_0^\infty x^2 e^{-x^2/2} dx$$

$u = \frac{2}{\sqrt{2\pi}} x$ $\qquad dv = x e^{-x^2/2} dx$

$du = 2/\sqrt{2\pi}\, dx$ $\qquad v = -e^{-x^2/2}$

$$\Rightarrow E(x^2) = \frac{-2}{\sqrt{2\pi}} x e^{-x^2/2} \Big|_0^\infty$$

$$+ \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/2} dx = 1$$

$$\Rightarrow \sigma^2 = 1 - \mu^2 = 1 - \frac{2}{\pi} = \frac{\pi - 2}{\pi}$$

(3-47) $X \sim n(5,10)$

$Pr[0.04 < (x-5)^2 < 38.4]$

$= Pr[0.004 < \left(\frac{x-5}{\sqrt{10}}\right)^2 < 3.84]$

$= Pr[0.004 < \chi^2(1) < 3.84]$

$= Pr[\chi^2(1) < 3.84] - Pr[\chi^2_1 < 0.004]$

$= 0.95 - 0.050$

$= 0.9$

(3-48) $X \sim n(1, 4)$

$$P_r[1 < x^2 < 9]$$
$$= 2P_r[1 < x < 3]$$
$$= 2P_r[0 < z < 1]$$
$$= 2[0.841 - 0.5]$$
$$= 2[0.341]$$
$$= 0.682$$

(3-50)

$$E[x^{2m}] = \frac{2m!}{2^m m!}$$

$$E[x^{2m-1}] = 0$$

$$M(t) = \sum_{m=0}^{\infty} \frac{E[x^m]\, t^m}{m!}$$

$$= \sum_{2m=0}^{\infty} \frac{\cancel{2m!}}{2^m m!} \frac{t^{2m}}{\cancel{(2m)!}}$$

$$= \sum_{2m=0}^{\infty} \frac{1}{m!} \left(\frac{t^2}{2}\right)^m$$

$$= e^{+t^2/2}$$

$$\Rightarrow n(0,1)$$

(3-51)

$X_1 \sim n(0,1)$

$X_2 \sim n(2,4)$

$X_3 \sim n(-1,1)$ $\quad (Y = \# < 0)$

$p_1 = \frac{1}{2}$

$p_2 = P[X_2 < 0] = P[Z < \frac{-2}{2}]$

$= P[Z < -1]$

$= 1 - P[Z < 1] = 1 - 0.841 = 0.159$

$p_3 = P[Z < 1] = 0.841$

$p(x_1 x_2 x_3) = n(0,1)\, n(2,4)\, n(-1,1)$

$P[Y=2] = (.5)(.159)^2 + (.5)(.841)^2$

$+ (.5)(.841)(.159)$

$= (.5)[(.159)^2 + (.841)^2 + (.841)(.159)]$

$= 0.433$

(3.53)

$X \sim n(0,1)$

$Y \sim n(0,1)$

$Z = X + Y$

$F(z) = P_r[Z \le z]$

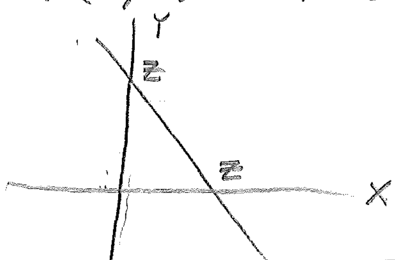
$= P_r[X + Y \le z]$

$\cancel{= P_r[X \le z - y \mid Y = y]\, P_r[Y = y]}$

$\cancel{= \int_{-\infty}^{z-y} f(x)\,dx\; f(y)}$

$\cancel{= \int_{-\infty}^{z-y} f(x) f(y)\,dx}$

$f(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$



$$F(z) = P_r[Z \le z] = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}\, dy\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_{y=-\infty}^{z-x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} F(z-x)\, dx$$

$$f(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} f(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-x)^2}{2}}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(2x^2 - 2xz)}{2}}\, dx$$

$$\int_{-\infty}^{\infty} e^{-(x^2 - xz)}\, dx$$

$$= \int e^{-(x-\frac{z}{2})^2} e^{\frac{z^2}{4}}\, dx = \sqrt{2\pi}\, e^{z^2/4}$$

$$\Rightarrow f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/4} \Rightarrow n(0,2)$$

(3-54) $\mu_1 = 3$ $\quad \sigma_2^2 = 25$

$\mu_2 = 1$ $\quad \rho = 3/5$

$\sigma_1^2 = 16$

(4-1) $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$

$= \frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + \bar{x}^2\right]$

$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2\bar{x}^2 + \frac{1}{n}\sum_{i=1}^{n}\bar{x}^2$

$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2\bar{x}^2 + \frac{\bar{x}^2}{\cancel{n}}\,\cancel{n}$

$\cancel{= \frac{1}{n}\Big[}\sum_{i=1}^{n} x_i^2 - 2\bar{x}^2\Big] + $

$= -\bar{x}^2 + \frac{1}{n}\sum_{i=1}^{n} x_i^2$

(4-3)

$f(x) = \frac{(x+1)}{2}$; $-1 < x < 1$

$P_r(X > 1) = \frac{1}{2}\int_0^1 (x+1)\,dx$

$= \frac{1}{2}\left[\frac{1}{2}x^2 + x\right]_0^1$

$= \frac{1}{2}\left[\frac{1}{2}1 + 1\right]$

$= \frac{1}{2}\ \frac{3}{2} = 3/4$

5 SAMPLES $\longrightarrow b(5, 3/4)$

$Y \sim \binom{5}{n}\left(\frac{3}{4}\right)^n\left(\frac{1}{4}\right)^{5-n} = f(n)$

$f(4) = \binom{5}{4}\left(\frac{3}{4}\right)^4 \frac{1}{4}$

$= 5 \quad \frac{81}{4^5}$

$= \frac{405}{1024}$

$$\begin{array}{r} 16 \\ \times 16 \\ \hline 96 \\ 16 \\ \hline 256 \\ \times 4 \\ \hline 1024 \end{array}$$

(4-4)

$$Y = Max[X_1, X_2 X_3]$$

$$Pr[Y > 8] = Pr[Max(X_1, X_2, X_3) > 8]$$
$$= 1 - Pr[Y < 8]$$

$$Pr[Y < 8] = Pr[Max(X_1, X_2, X_3) < 8]$$
$$= Pr[X_1 < 8, X_2 < 8, X_3 < 8]$$
$$= Pr[X_1 < 8] Pr[X_2 < 8] Pr[X_3 < 8]$$
$$= (Pr[X < 8])^3$$

$$X \sim n(6, 4)$$

$$Pr[X < 8] = Pr\left[\frac{X-6}{2} = Z < 1\right]$$
$$Pr[Z < 1] = 0.841$$

$$Pr[Y < 8] = (0.841)^3 = 0.594$$

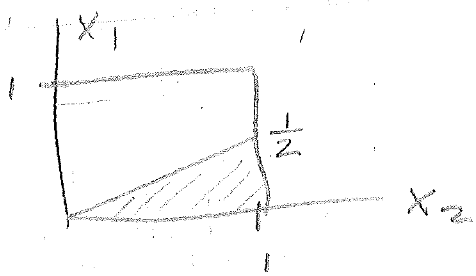
$$\therefore Pr[Y > 8] = 1 - 0.594 = 0.405$$

(4-5) X_1 AND $X_2 \sim f(x)$

$$f(x) = \begin{cases} 2x & ;\ 0<x<1 \\ 0 & ;\ \text{OTHERWISE} \end{cases}$$

FIND $P_r[X_1/X_2 \le \frac{1}{2}]$

$= P_r[X_1 \le \frac{1}{2} X_2]$



$$f(x_1, x_2) = \begin{cases} 4 x_1 x_2 & ;\ 0<x_1, x_2<1 \\ 0 & \text{OTERWISE} \end{cases}$$

$$P_r[X_1 \le \tfrac{1}{2} X_2] = \int_0^{\frac{1}{2}} \int_{x_1=0}^{\frac{1}{2}x_2} f \, dx_1 \, dx_2$$

$$= \int_0^{\frac{1}{2}} \int_{x_1=0}^{\frac{1}{2}x_2} 4x_1 x_2 \, dx_1 \, dx_2$$

$$= \int_0^{\frac{1}{2}} 4x_2 \, \tfrac{1}{2} x_1^2 \Big|_0^{\frac{1}{2}x_2} dx_2$$

$$= \int_0^{\frac{1}{2}} 2x_2 \left[\tfrac{1}{4} X_2^2\right] dX_2$$

$$= \int_0^{\frac{1}{2}} \tfrac{1}{2} x_2^3 \, dx_2 =$$

$$= \tfrac{1}{8} X_2^4 \Big|_0^{\frac{1}{2}}$$

$$= \frac{1/16}{1/8} \cancel{= \tfrac{1}{2}} \qquad \frac{1}{8}$$

(4-6) FOR $n = 2$

$$s^2 = \frac{x_1^2 + x_2^2}{2} - \frac{(x_1 + x_2)^2}{4}$$

$$= \frac{1}{4}\left[2x_1^2 + 2x_2^2 - x_1^2 - 2x_1x_2 - x_2^2\right]$$

$$= \frac{1}{4}\left[x_1 - x_2\right]^2$$

$$\Longrightarrow C = \frac{1}{4}$$

(4-7) $x_i = i \quad ; i = 1, 2, \ldots, n$

$$\bar{x} = \frac{1}{n} \sum_i x_i = \frac{1}{n} \sum_{i=1}^{n} i$$

$$= \frac{1}{n} \cdot \frac{n(n+1)}{2}$$

$$= \frac{(n+1)}{2}$$

$$\overline{x^2} = \frac{1}{n} \sum_i x_i^2$$

$$= \frac{1}{n} \sum i^2 = \frac{1}{n}\left[\frac{1}{6} n(n+1)(2n+1)\right]$$

$$= \frac{1}{6}(n+1)(2n+1)$$

$$s^2 = \frac{1}{6}(n+1)(2n+1) - \frac{1}{4}(n+1)^2$$

$$= \frac{1}{24}\left[4(n+1)(2n+1) - 6(n+1)^2\right]$$

$$= \frac{1}{24}\left[8n^2 + 4n + 8n + 4 - 6n^2 - 12n - 6\right]$$

$$= \frac{1}{24}\left[2n^2 - 2\right]$$

$$= \frac{1}{12}\left[n^2 - 1\right]$$

(4-8) $y_i = a + b x_i$

$i = 1, 2, \ldots, n$

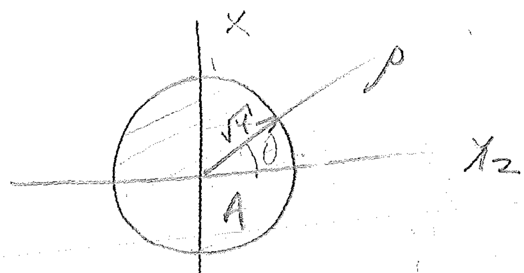
$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n}(a + b x_i)$$
$$= \frac{1}{n}\left[na + b\sum_i x_i\right]$$
$$= a + b\bar{x}$$
$$\overline{Y^2} = \frac{1}{n}\sum_{i=1}^{n}(a + b x_i^2)^2$$

$$= \frac{1}{n}\left[\sum a^2 + 2ab x_i + b^2 x_i^2\right]$$
$$= \frac{1}{n}\left[na^2 + 2ab\sum x_i + b^2\sum x_i^2\right]$$
$$= a^2 + 2ab\bar{x} + b^2\overline{x^2}$$
$$\Rightarrow s^2 = \overline{Y^2} - \bar{Y}^2$$
$$= a^2 + 2ab\bar{x} + b^2\overline{x^2} - (a + b\bar{x})^2$$
$$= \cancel{a^2} + 2ab\bar{x} + b^2\overline{x^2} - \cancel{a^2} - 2ab\bar{x} - b^2\bar{x}^2$$
$$= \cancel{2ab\bar{x}} + b^2\overline{x^2} - \cancel{2ab\bar{x}} - b^2\bar{x}^2$$
$$= b^2\overline{x^2} - b^2\bar{x}^2$$
$$= b^2(\overline{x^2} - \bar{x}^2)$$
$$= b^2 s_x^2$$

(4-9) $Y = X_1^2 + X_2^2$

$F(Y) = P_r[Y < Y]$

$= P_r[X^2 + X_2^2 < Y]$



$$F(Y) = \iint_A \frac{1}{2\pi} e^{-\left(\frac{x^2 + x_1^2}{2}\right)} dx_1 dx_2$$

$x_1 = \rho \sin\theta \qquad x_2 = \rho \cos\theta$

$$F(Y) = \int_0^{\sqrt{Y}} \int_0^{2\pi} \frac{1}{2\pi} e^{-\rho^2/2} \rho \, d\rho \, d\theta$$

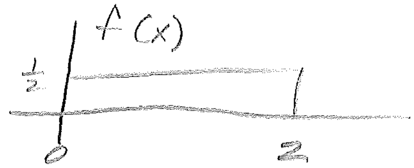
$$= \int_0^{\sqrt{Y}} e^{-\rho^2/2} \rho \, d\rho$$

$$= -e^{-\rho^2/2} \Big|_0^{\sqrt{Y}} = -e^{-\frac{Y}{2}} + 1$$

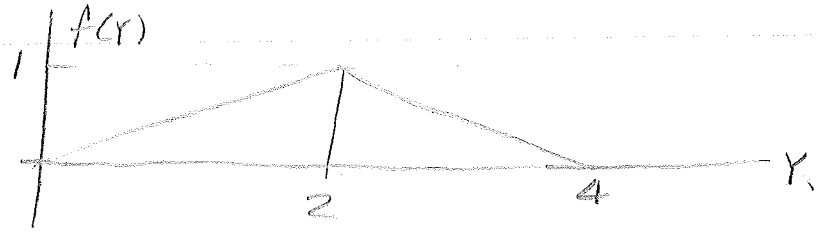
$$= 1 - e^{-Y/2}$$

$$f(Y) = \frac{1}{2} e^{-Y/2} = \chi^2(2)$$

(4-10) X_1, X_2



$Y = X_1 + X_2$



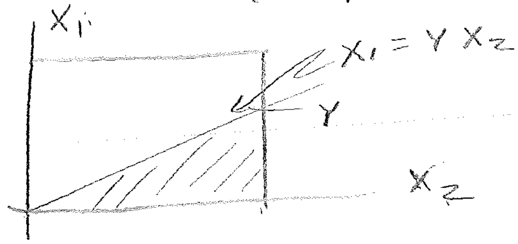
$$f(Y) = \begin{cases} \frac{1}{2} Y & ; \; 0 < Y < 2 \\ -\frac{1}{2} Y + 2 & ; \; 2 < Y < 4 \\ 0 & \text{OTHERW} \end{cases}$$

(4.11) $f(x_i) = 1 \; ; 0 < x < 1$

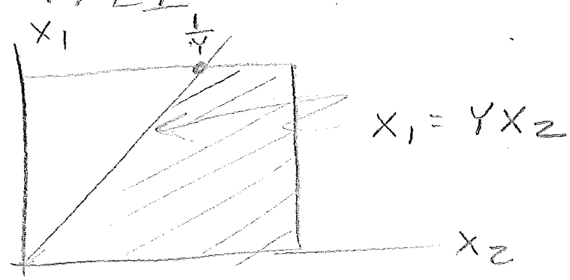
$f(x_1, x_2) = 1 \quad ; \; 0 < x_1, x_2 < 1$

$Y = X_1 / X_2$

$\Rightarrow X_1 = Y X_2$



$P_r[\underline{Y} < Y] = \frac{1}{2} Y \quad ; \; 0 < Y < 1$



$P_r[\underline{Y} < Y] = 1 - \frac{1}{2Y} \quad ; \; 1 < Y < \infty$

THUS

$$F(Y) = \begin{cases} \frac{1}{2} Y & ; \; 0 < Y < 1 \\ 1 - \frac{1}{2Y} & ; \; 1 < Y < \infty \end{cases}$$

$$f(Y) = \begin{cases} \frac{1}{2} & ; \; 0 < Y < 1 \\ \frac{1}{2Y^2} & ; \; 1 < Y < \infty \end{cases}$$

(4-12) $f(x) = 5x^4 \quad ; 0<x<1$

$f(x_1 x_2 x_3) = 5^3 x_1^4 x_2^4 x_3^4 \quad ; 0 < x_1 x_2 x_3 < 1$

$$Pr[\underline{X} < Y] = Pr[X_1 < Y]\, Pr[X_2 < Y]\, Pr[X_3 < Y]$$

$$= \{Pr[X < Y]\}^3$$

$$= \left[\int_0^Y 5x^4 dx\right]^3$$

$$= Y^{15} \quad ; 0 < Y < 1$$

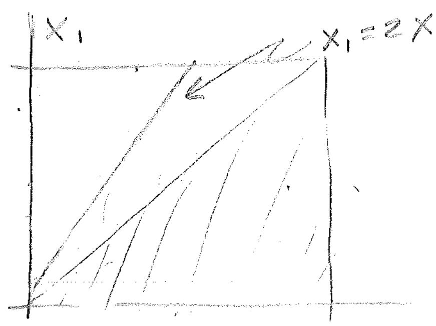
$$f(Y) = \begin{cases} 15Y^{14} & ; 0<Y<1 \\ 0 & \text{ELSE} \end{cases}$$

(4.13) $f(x) = 2x \; ; 0 < x < 1$

FIND
$Pr[X_1 < X_2 | X_1 < 2X_2]$

$$= \frac{Pr[X_1 < X_2, X_1 < 2X_2]}{Pr[X_1 < 2X_2]}$$

$f(x_1, x_2) = 4x_1x_2 \quad 0 < x_1 x_2 < 1$



$0 < x_1 < x_2$

$Pr[X_1 < X_2, X_1 < 2X_2] = Pr[X_1 < X_2]$

$$= \int_0^1 4x_2 \int_{x_1=0}^{x_2} x_1 \, dx_1 \, dx_2$$
$$= \int_0 4x_2 \, \tfrac{1}{2} x_2^2 \, dx_2$$
$$= \int_0^1 2x_1^3 \, dx_1 = \frac{2}{4} x_1^4 \Big|_0^1 = \frac{1}{2}$$

$Pr[X_1 < 2X_2]$
$$= 1 - \int_{x_1=0}^{1} 4x_1 \int_{x_2=0}^{\frac{x_1}{2}} x_2 \, dx_2 \, dx_1$$
$$= 1 - 4 \int_{x_1=0}^{1} x_1 \, \tfrac{1}{2} x_2^2 \Big|_0^{x_1/4} dx_1$$
$$= 1 - 2 \int_0^1 x_1 \, \frac{x_1^2}{16}$$
$$= 1 - \frac{1}{8} \int_0^1 x_1^3 \, dx$$
$$= 1 - \frac{1}{32} = 7/32$$

$\Rightarrow Pr[X_1 < X_2 | X_1 < 2X_2]$

$$= \frac{1/2}{7/32} = \frac{16}{7}$$

WRONG!!

(4-14) $f(x) = \frac{1}{3}$; $x = 1, 2, 3$

$f(y) = \frac{1}{3}$; $y = 3, 5, 7$

(4-15)

$$f(x_1, x_2) = \left(\frac{2}{3}\right)^{x_1 + x_2} \left(\frac{1}{3}\right)^{2 - x_1 - x_2}$$

$$(x_1, x_2) = (0,0), (0,1), (1,0), (1,1)$$

$$\begin{cases} Y_1 = X_1 - X_2 \\ Y_2 = X_1 + X_2 \end{cases}$$

$$2X_1 = Y_1 + Y_2 \Rightarrow X_1 = \frac{Y_2 + Y_1}{2}$$

$$Y_2 - Y_1 = 2X_2 \Rightarrow X_2 = \frac{Y_2 - Y_1}{2}$$

$$f(Y_1, Y_2) = \left(\frac{2}{3}\right)^{Y_2} \left(\frac{1}{3}\right)^{2 - Y_2}$$

$$(Y_1, Y_2) = (0,0), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right), (1,0)$$

(4-16) $f(x) = \left(\frac{1}{2}\right)^x \; ; x = 1, 2, 3, \ldots$

$Y = X^3$

$\Rightarrow X = + Y^{\frac{1}{3}}$

$f(y) = \left(\frac{1}{2}\right)^{y^{\frac{1}{3}}} \; ; Y = 1, 2^3, 3^3, \ldots$

(4-17) $f(x_1, x_2) = \dfrac{x_1 x_2}{36}$ $\quad x_1 = 1, 2, 3 = x_2$

$Y_1 = X_1 X_2 \qquad Y_2 = X_2$

$X_1 = Y_1 / Y_2 \qquad X_2 = Y_2$

$Y_1 = \underset{(1,1)}{\underline{1}}, \underset{(12)}{\underline{2}}, \underset{(13)}{\underline{3}}, \underset{(21)}{\underline{2}}, \underset{(22)}{\underline{4}}, \underset{(23)}{\underline{6}}, \underset{(31)}{\underline{3}}, \underset{(32)}{\underline{6}}, \underset{(33)}{9}$

$$f(Y_1) = \begin{cases} 1 & ; \ \frac{1}{36} \\ 2 & ; \ \frac{4}{36} \\ 3 & ; \ \frac{6}{36} \\ 4 & ; \ \frac{4}{36} \\ 6 & ; \ \frac{12}{36} \\ 9 & \ \ \ \frac{9}{36} \end{cases}$$

$X_2 \sim b(n_2, p)$

(4-18) $X_1 \sim b(n_1, p)$

$f(x_1) = \binom{n}{x_1} p^{x_1} (1-p)^{x_1}$

$f(x_2) = \binom{n}{x_2} p^{x_2} (1-p)^{x_2}$

$Y_1 = X_1 + X_2$ $\quad Y_2 = X_2$

$X_1 = Y_1 - Y_2$ $\quad X_2 = Y_2$

$$f(x_1, x_2) = \binom{n}{x_1}\binom{n}{x_2} p^{x_1+x_2} (1-p)^{x_1+x_2}$$

$$f(Y_1, Y_2) = \binom{n}{Y_2}\binom{n}{Y_1 - Y_2} p^{Y_1} (1-p)^{Y_1}$$

FROM HINT:

$$f(Y_1) = \sum_{Y_2} f(Y_1, Y_2) = \binom{2n}{Y_1} p^{Y_1} (1-p)^{Y_1}$$

$$= b(2n, p)$$

(4-19)

$X_1, X_2, \text{\&} X_3$ ARE POISSON

$Y = X_1 + X_2 + X_3$

$M(t) = e^{\mu[e^t - 1]}$

ADDITION $\Longrightarrow$ CONVOLUTION

$\Longrightarrow$ MULTIPICATION

$$M_Y(t) = \left[M(t)\right]^3$$

$$= e^{3\mu[e^t - 1]}$$

$$\Longrightarrow Y \sim \text{POIS}(3\mu)$$

(4·20)

$$f(x_1, x_2) = f_1(x_1) f_2(x_2)$$

$$Y_1 = u_1(x_1) \qquad Y_2 = u_2(x_2)$$

$$x_1 = u_1^{-1}(Y_1) \qquad x_2 = u_2^{-1}(Y_2)$$

$$P_r[\mathbb{X}_1 = u_1(x_1),\ \mathbb{X}_2 = u_2(x_2)]$$

$$= P_r[\mathbb{X}_1 = x_1\ \mathbb{X}_2 = x_2] \Longleftarrow \text{SEPARABLE}$$

(4-21) $f(x) = \frac{x^2}{9}$; $0 < x < 3$

$Y = X^3 \Rightarrow X = Y^{\frac{1}{3}}$

$$f(Y) = \frac{\left(Y^{\frac{1}{3}}\right)^2}{9} \frac{d}{dY} Y^{\frac{1}{3}}$$

$$= \frac{1}{9} Y^{\frac{2}{3}} \ \frac{1}{3} Y^{-\frac{2}{3}}$$

$$= \frac{1}{27} \qquad ; \ 0 < x < 27$$

(4.22) $f(x) = 2xe^{-x^2}$; $0 < x < \infty$

$Y = X^2 \Rightarrow X = Y^{\frac{1}{2}}$

$$f(Y) = 2\,(Y^{\frac{1}{2}})\,e^{-(Y^{\frac{1}{2}})^2} \frac{d}{dY} Y^{\frac{1}{2}} \; ; 0 < Y < \infty$$

$$= 2\sqrt{Y}\, e^{-Y} \, \frac{1}{2} \, Y^{-\frac{1}{2}}$$

$$= e^{-Y} \; ; 0 < Y < \infty$$

(4-23)(a) $Y = X^2$

$F(Y) = P_r[\underline{Y} \le Y] = P_r[\underline{X}^2 \le Y]$

$= P_r[\sqrt{Y} < \underline{X} < \sqrt{Y}]$

$= \int_{\sqrt{Y}}^{\sqrt{Y}} f_x(x)\,dx$

FROM LEIBNITZ:

$f(Y) = f_x(\sqrt{Y})\frac{d}{dY}\sqrt{Y} - f_x(-\sqrt{Y})\frac{d}{dY}(-\sqrt{Y})$

$= [f_x(\sqrt{Y}) + f_x(-\sqrt{Y})]\frac{1}{2\sqrt{Y}}$

(b) $f(x) = +f(-x)$

$\Rightarrow f(Y) = \frac{1}{\sqrt{Y}} f_x(\sqrt{Y})$

(c) $f(X) = 0 \quad \forall \; X < 0$

$\Rightarrow f_Y = \frac{1}{2\sqrt{Y}} f_x(\sqrt{Y})$

(4-24) $Y_1 = X_1/X_2 \qquad Y_2 = X_2$

$X_1 = Y_2 Y_1 \qquad X_2 = Y_2$

$$J = \begin{vmatrix} Y_2 & Y_1 \\ 0 & 1 \end{vmatrix} = |Y_2|$$

$$f(x_1, x_2) = \frac{1}{2\pi} e^{-\frac{(x_1^2 + x_2^2)}{2}}$$

$$f(Y_1 Y_2) = \frac{|Y_2|}{2\pi} e^{-[(Y_2 Y_1)^2 + Y_2^2]/2}$$

$$= \frac{|Y_2|}{2\pi} e^{-Y_2^2(Y_1^2+1)/2}$$

$$f(Y_1) = \int_{Y_2} f(Y_1 Y_2)\, dY_2$$

$$= \frac{1}{2\pi} \times 2 \int_0^\infty Y_2 e^{-Y_2^2(Y_1^2+1)/2} dY_2$$

$$= \frac{2}{2\pi} \frac{-1}{Y_1^2+1} e^{-Y_2^2(Y_1^2+1)/2}\Big|_0^\infty$$

$$= \frac{1}{\pi} \frac{-1}{Y_1^2+1} (0-1)$$

$$= \frac{1}{\pi(Y_1^2+1)}$$

(4-25)

$$f(x_1 x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x_1^{\alpha-1} x_2^{\beta-1} e^{-x_1-x_2}$$

$$Y_1 = \frac{x_1}{x_1+x_2} \qquad Y_2 = x_1 + x_2 = \frac{x_1}{Y_2}$$

$$x_1 = Y_1 Y_2 \qquad x_2 = Y_2 - Y_1 Y_2$$

$$J = \begin{vmatrix} Y_2 & Y_1 \\ -Y_2 & 1-Y_1 \end{vmatrix} = Y_2(1-Y_1) + Y_1 Y_2 = |Y_2|$$

$$f(Y_1 Y_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (Y_1 Y_2)^{\alpha-1} |Y_2| [Y_2 - Y_1 Y_2]^{\beta-1} e^{-Y_1Y_2 - Y_2 + x_2 Y_2}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} Y_1^{\alpha-1} |Y_2| Y_2^{\alpha-1} Y_2^{\beta-1} (1-Y_1)^{\beta-1} e^{-Y_2}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} Y_1^{\alpha-1} |Y_2| (1-Y_1)^{\beta-1} Y_2^{\alpha+\beta-2} e^{-Y_2}$$

$$f_1(Y_1) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} Y_1^{\alpha-1} (1-Y_1)^{\beta-1}$$

$$\underbrace{\int_0^\infty Y_2^{\alpha+\beta-1} e^{-Y_2} dY_2}_{=\Gamma(\alpha+\beta)}$$

$$\Rightarrow f_1(Y_1) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} Y_1^{\alpha-1} (1-Y_1)^{\beta-1}$$

(4-26)

$$g(Y) = \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}_{C} Y^{\alpha-1}(1-Y)^{\beta-1}; \quad 0<Y_i<1$$

$$E[Y_i] = C\int_0^1 Y^{\alpha}(1-Y)^{\beta-1}\,dY$$

$$= C\,\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)}\,\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}$$

BUT $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$

$$\Rightarrow \bar{X} = \frac{\alpha}{\alpha+\beta}$$

$$E[Y^2] = C\int_0^1 Y^{\alpha+1}(1-Y)^{\beta-1}\,dY$$

$$= C\,\frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)}$$

$$= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}\,\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+2)}$$

$$= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$$

$$\sigma^2 = E[Y^2] - \bar{Y}^2$$

(4-27) $\Gamma(n) = (n-1)!$

(a) $f(x) = cx(1-x)^3$

$\alpha = 2 \quad \beta = 4$

$\Gamma(\alpha+\beta) = 5!$

$\Rightarrow C = \frac{5!}{3!} = 20$

(b) $f(x) = cx^4(1-x)^5$

$\alpha = 5 \quad \beta = 6$

$C = \frac{10!}{4!\ 5!} = \frac{10 \cdot 9 \cdot \cancel{8} \cdot 7 \cdot \cancel{6}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}$

$= 1260$

$\begin{array}{r} 14 \\ 90 \\ \hline 1260 \end{array}$

(c) $f(x) = cx^2(1-x)^8$

$\alpha = 3 \quad \beta = 9$

$C = \frac{11!}{2!\ 8!}$

$= \frac{11 \cdot \cancel{10} \cdot 9}{\cancel{2} \cdot 1}$

$\begin{array}{r} 55 \\ 9 \\ \hline 495 \end{array}$

$= 495$

(4-28)

$$f(x) = C\,x\,(3-x)^4 \quad ; \; 0 < x < 3$$

$$Y = X/3 \Rightarrow X = 3Y$$

$$\frac{dx}{dY} = 3$$

$$f(Y) = 3C\,(3Y)\,(3-3Y)^4$$

$$= 3^6 C \; Y((1-Y)^4)$$

$$\alpha = 2 \qquad \beta = 5$$

$$3^6 C = \frac{6!}{1!\;4!} = \quad 6 \cdot 5 = 30$$

$$\Rightarrow C = \frac{30}{3^6} = \frac{10}{3^5}$$

$$3^5 = 243$$

$$\Rightarrow C = {}^{10}/_{243}$$

(4-29)

$$g(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} Y^{\alpha-1}(1-Y)^{\beta-1}$$

$$\alpha = \beta$$

$$g(x) = \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} Y^{\alpha-1}(1-Y)^{\alpha-1}$$

$$= \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} (Y-Y^2)^{\alpha-1}$$

$$g(Y-\tfrac{1}{2}) = \frac{\Gamma(\alpha)}{\Gamma^2(\alpha)} (Y-\tfrac{1}{2})^{\alpha-1}(\tfrac{3}{2}-Y)^{\alpha-1}$$

$1-Y+\frac{1}{2}$

$$g(Y) = \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} [Y(1-Y)]^{\alpha-1}$$

$$g(Y-\tfrac{1}{2}) = \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} [(Y-\tfrac{1}{2})(\tfrac{3}{2}-Y)]^{\alpha-1}$$

$$= \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} [Y^2+\tfrac{1}{2}Y-\tfrac{3}{4}]^{\alpha-1}$$

(4.30) $X \sim U(-\pi/2, \pi/2)$

$$f(x) = \begin{cases} \frac{1}{\pi} & -\pi/2 < x < \pi/2 \\ 0 & \text{OTHER} \end{cases}$$

$Y = \tan x$

$X = \tan^{-1}(Y) \Rightarrow \frac{dx}{dY} = \frac{1}{Y^2+1}$

$$f(Y) = \begin{cases} \frac{1}{\pi} \frac{1}{Y^2+1} & ; -\infty < Y < \infty \end{cases}$$

(4-31) $f(x) = \frac{(x-1)}{2} \quad ; \quad 1 < x < 3$

$Y = U(x) \quad x = w(Y)$

$$\left.\begin{array}{l} 1 = w(0) \\ 3 = w(1) \end{array}\right\} \text{BOUNDRY CONDITIONS}$$

$$f(Y) = \frac{w(Y)-1}{2} \; \frac{dw(Y)}{dY} = 1$$

$$\Rightarrow \frac{dw}{dY} = \frac{2}{w-1}$$

$$dw(w-1) = 2\,dY$$

$$\frac{1}{2}w^2 - w = 2Y + C$$

$$w^2 - 2w - 4(Y+C) = 0$$

$$w = \frac{2 \pm \sqrt{4 + 16(Y+C)}}{2}$$

$$= 1 \pm \sqrt{1 + 4(Y+C)}$$

$$w(0) = 1 = 1 \pm \sqrt{1 + 4C} \quad \Rightarrow C = -\frac{1}{4}$$

$$w = 1 \pm \sqrt{1 + 4(Y - \tfrac{1}{4})}$$

$$= 1 \pm \sqrt{Y}$$

(4-32)

$f(x) = \frac{1}{9} \quad ; \quad 1 < x < 10$

$Y_1 = X_1 X_2 \qquad Y_2 = X_2$

$X_1 = Y_1 / Y_2 \qquad X_2 = Y_2$

$$J = \begin{bmatrix} \frac{1}{Y_2} & \frac{-Y_1}{Y_2^2} \\ 0 & 1 \end{bmatrix} = \frac{1}{Y_2}$$

$f(x_1, x_1) = \frac{1}{9} \quad 1 < x_1 x_2 < 10$

$f(Y_1 Y_2) = \frac{1}{9Y_2} \quad ; \quad 1 < Y_1 < 100$

$1 < Y_2 < 10$

$$f(Y_1) = \frac{1}{9} \int_1^{10} \frac{1}{Y_2} \, dY_2$$

$$= \frac{1}{9} \quad \frac{-1}{Y^2} \Big|_1^{10}$$

$$= \frac{1}{9} \left[\frac{-1}{100} + 1 \right]$$

$$= \frac{99}{900} \quad 1 < Y_1 < 100$$

(4-31);

$f(x) = \frac{x-1}{2} \qquad 1 < x < 3$

$\frac{x-1}{2} \left|\frac{dx}{dY}\right| = 1$

ASSUME $\frac{dx}{dY} > 0$

$\Rightarrow \frac{x-1}{2} \frac{dx}{dY} = 1 \Rightarrow \frac{x-1}{2} dx = dY$

$\frac{1}{2}\left(\frac{x^2}{2} - x\right) = Y + C$

$Y = U(x) = \frac{1}{2}\left(\frac{x^2}{2} - x\right) + C$

$U(\text{END}) = 1$

$U(\text{OTHER END}) = 0$

ASSUME $(1, 3) \longrightarrow (0, 1)$

AND IS NONDECREASING

$U(1) = 0 = \frac{1}{2}\left(\frac{1}{2} - 1\right) + C \Rightarrow C = \frac{1}{4}$

AND $Y = \frac{1}{2}\left(\frac{x^2}{2} - x\right) + \frac{1}{4} = \frac{(x-1)^2}{4}$

NOTE $U(3) = 1$

$Pr(a \le Y \le b) = b - a$

$Pr[a \le U(x) \le b]$

$= Pr[U^{-1}(a) < x \le U^{-1}(b)]$

$= F[U^{-1}(b)] - F[U^{-1}(a)] = b - a$

$\Rightarrow U = F^{-1}$

$\Uparrow$

PROBABILITY INTEGRAL XFORM

(4-33) $X_1, X_2 \sim n(\mu, \sigma^2)$

$Y_1 = X_1 + X_2 \qquad Y_2 = X_1 - X_2$

$X_1 = \dfrac{Y_1 + Y_2}{2} \qquad X_2 = \dfrac{Y_1 - Y_2}{2}$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}$$

$$f(x_1, x_2) = \frac{1}{2\pi\sigma} e^{-\frac{1}{2\sigma^2}\left[(x_1-\mu)^2 + (x_2-\mu)^2\right]}$$

$$f(Y_1, Y_2) = \frac{1}{4\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left[\left(\frac{Y_1+Y_2}{2}-\mu\right)^2 + \left(\frac{Y_1-Y_2}{2}-\mu\right)^2\right]}$$

$$= \frac{1}{4\pi\sigma^2} e^{-\frac{1}{8\sigma^2}\left[(Y_1+Y_2-2\mu)^2 + (Y_1-Y_2-2\mu)^2\right]}$$

$$= \frac{1}{4\pi\sigma^2} e^{-\frac{1}{8\sigma^2}\left[(Y_1+Y_2)^2 + (Y_1-Y_2)^2 + 8\mu^2 - 4\mu(Y_1 + \cancel{Y_2}) - 4\mu(Y_1 - \cancel{Y_2})\right]}$$

$$= \frac{1}{4\pi\sigma^2} e^{-\frac{1}{8\sigma^2}\left[2Y_1^2 + 2Y_2^2 + 8\mu^2 - 8\mu Y_1\right]}$$

$$= \frac{1}{4\pi\sigma^2} e^{-\frac{1}{4\sigma^2}\left[Y_1^2 + Y_2^2 + 4\mu^2 - 4\mu Y_1\right]}$$

$$= \frac{1}{4\pi\sigma^2} e^{-\frac{1}{4\sigma^2}\left[(Y_1-2\mu)^2 + Y_2^2\right]}$$

$\Rightarrow$ BIV NORMAL $\quad \sigma_{Y_1}^2 = \sigma_{Y_2}^2 = 2\sigma^2$

$\mu_{Y_1} = 2\mu \quad \mu_{Y_2} = 0$

(4-34)

$X_1, X_2 \sim n(\mu, \sigma^2)$

$Y_1 = X_1 + X_2 \qquad Y_2 = X_1 + 2X_2$

$$f(x_1 x_2) = \frac{1}{2\pi}\frac{1}{\sigma^2} e^{-\frac{1}{2}\left[\frac{(x_1-\mu)^2}{\sigma^2} + \frac{(x_2-\mu)^2}{\sigma^2}\right]}$$

$X_1 = 2Y_1 - Y_2 \qquad X_2 = Y_2 - Y_1$

$$J = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow |J| = 1$$

THUS

$$f(Y_1 Y_2) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left[(2Y_1 - Y_2 - \mu)^2 + (Y_2 - Y_1 - \mu)^2\right]}$$

$E[Y_1] = 2\mu = \mu_1 \qquad E(Y_2) = 3\mu = \mu_2$

$$= \left(\frac{1}{2\pi\sigma^2}\right) e^{-\frac{1}{2\sigma^2}\left[\{2(Y_1-\mu_1) - (Y_2-\mu_2) - \mu + \mu\}^2 + \{(Y_2 - Y_2) - (Y_1 - U_1)\}^2\right.}$$

ETC.

(4-35) $T \sim t_{10}$

$$\begin{aligned}
&P_r[\,|T| > 2.228\,] \\
&= 1 - 2\,P_r[\,0 < T < 2.228\,] \\
&= 1 - 2\left[\,P_r[\,T < 2.228\,] - \tfrac{1}{2}\,\right] \\
&= 1 - 2\,[\,0.975 - .5\,] \\
&= 1 - 2\,[\,0.475\,] \neq 0.950 = 0.005
\end{aligned}$$

4.36. FIND $b \ni$

$$Pr[-b < T < b] = 0.9$$

$$Pr[-b < T < b] = 2\left[Pr[T < b] - \tfrac{1}{2}\right]$$
$$= 2\,Pr[T < b] - 1$$

$$\Rightarrow Pr[T < b] = 0.95$$
$$\Rightarrow b = 1.761$$

(4.37) $\quad f \sim F$

$$g(f) = \frac{\Gamma\left(\frac{r_1+r_2}{2}\right)\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}}}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)} \frac{f^{\frac{r_1}{2}-1}}{(1-r_1 f/r_2)^{(r_1+r_2)/2}}$$

$$F = \frac{U/r_1}{V/r_2}$$

$$\Rightarrow \frac{1}{F} = \frac{V/r_2}{U/r_1}$$

$\Rightarrow \frac{1}{F}$ IS DISTRIBUTED AS
F WITH r_2 & r_1 D.O.F.

(4-38) $F_{5,10}$ $\quad \frac{1}{F} \sim F_{10,5}$

$$Pr[F \le a] = 0.05$$

$$Pr\left[\frac{1}{F} \ge \frac{1}{a}\right] = 0.05$$

$$= 1 - Pr\left[\frac{1}{F} \le \frac{1}{a}\right]$$

$$Pr\left[\frac{1}{F} \le \frac{1}{a}\right] = 0.95$$

$$\Rightarrow \frac{1}{a} = \frac{1}{4.74} = 0.21$$

$$0.95 = Pr[F \le b] \Rightarrow b = \cancel{4.74}\ 3.33$$

$$\Rightarrow Pr[a < F < b] = \overset{3.33}{\cancel{4.74}} - 0.21$$

$$\begin{array}{r} .21 \\ \cancel{4.53} \end{array} \quad = 3.12$$

(4.39) $$T^2 = \frac{W^2}{\left(\sqrt{V/r}\right)^2} = \frac{W^2}{V/r}$$

$$W^2 \sim \chi^2(1)$$

$$\Rightarrow T^2 \sim \frac{\chi_1^2(1)/1}{\chi_r^2/r} = F_{1,r}$$

$$(4\text{-}40)\quad Y = \frac{1}{1+\frac{r_1}{r_2}F} \Rightarrow 1+\frac{r_1}{r_2}F = \frac{1}{Y}$$

$$F = \frac{r_2}{r_1}\left(\frac{1}{Y}-1\right)$$

$$|J| = \left|\frac{dF}{dY}\right| = \frac{r_2}{r_1 Y^2}$$

$$f_Y(Y) = f_F\left[\frac{r_2}{r_1}\left(\frac{1}{Y}-1\right)\right]\frac{r_2}{r_1 Y^2}$$

$$= \frac{r_2/r_1}{Y^2}\,\frac{\Gamma\left[\frac{r_1+r_2}{2}\right]}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)}\;\frac{\cancel{\left(\frac{r_2}{r_1}\right)^{\frac{r_1}{2}-1}}\;\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}}\left[\frac{1}{Y}-1\right]^{\frac{r_1}{2}-1}}{\left[1+\cancel{\frac{r_1}{r_2}}\left\{\cancel{\frac{r_2}{r_1}}\left(\frac{1}{Y}-1\right)\right\}\right]^{\frac{r_1+r_2}{2}}}$$

$$= \frac{\Gamma\left(\frac{r_1+r_2}{2}\right)}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)}\;\frac{\left[\frac{1}{Y}-1\right]^{\cancel{r_1/2-1}}}{Y^2\left(\frac{1}{Y}-1\right)^{\cancel{\frac{r_1+r_2}{2}}}}$$

$$= \frac{\Gamma}{\Gamma\,\Gamma}\;\frac{1}{Y^2\left(\frac{1}{Y}-1\right)^{\frac{r_2}{2}+1}}$$

$$= \frac{\Gamma}{\Gamma\,\Gamma}\;\frac{1}{Y^2\,Y^{-\left(\frac{r_2}{2}+1\right)}\left(1-Y\right)^{\frac{r_2}{2}+1}}$$

$$= \frac{\Gamma}{\Gamma\,\Gamma}\;\frac{Y^{\frac{r_2}{2}-1}}{(1-Y)^{\frac{r_2}{2}+1}}$$

$$= \frac{\Gamma\left(\frac{r_1+r_2}{2}\right)}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)}\,Y^{\frac{r_2}{2}-1}\left(1-Y\right)^{-\frac{r_2}{2}-1}$$

(4-41) $f(x_1 x_2) = e^{-x_1 - x_2}$

$z_1 = \frac{x_1}{x_2} \qquad z_2 = x_2$

$x_1 = z_1 z_2 \qquad x_2 = z_2$

$J = \begin{vmatrix} z_2 & z_1 \\ 0 & 1 \end{vmatrix} = z_2$

$f(z_1 z_2) = z_2 e^{-z_1 z_2 - z_2}$

$= z_2 e^{-z_2(1+z_1)}$

$f(z_1) = \int_0^\infty z_2 e^{-z_2(1+z_1)} dz_2$

$u = z_2 \qquad dv = e^{-z_2(1+z_1)}$

$du = dz_2 \qquad v = \frac{-1}{1+z_1} e^{-z_2(1+z_1)}$

$f(z_1) = \underbrace{\frac{-z_2}{1+z_1} e^{-z_2(1+z_1)} \Big|_0^\infty}_{0}$

$+ \frac{1}{1+z_1} \int_0^\infty e^{-z_2(1+z_1)} dz_2$

$= \frac{1}{(1+z_1)^2}$

TRY

$r_1 = 2 \qquad r_2 = 2$

$\Rightarrow \Gamma\left(\frac{r_1+r_2}{2}\right) = \Gamma\left(\frac{r_1}{2}\right) = \Gamma\left(\frac{r_2}{2}\right) = 1$

YUP!

$z_1 \sim F_{2,2}$

(4·42) $x_1 = Y_1 \cos Y_2 \sin Y_3$

$X_2 = Y_1 \sin Y_2 \sin Y_3$

$X_3 = Y_1 \cos Y_3$

$$J = \begin{vmatrix} \cos Y_2 \sin Y_3 & -Y_1 \sin Y_2 \sin Y_3 & Y_1 \cos Y_2 \cos Y_3 \\ \sin Y_2 \sin Y_3 & Y_1 \cos Y_2 \sin Y_3 & Y_1 \sin Y_2 \cos Y_3 \\ \cos Y_3 & 0 & -Y_1 \sin Y_3 \end{vmatrix}$$

$$= \left| -\left(Y_1^2 \cos^2 Y_2 \sin^3 Y_3 + Y_1^2 \sin^2 Y_2 \cos^3 Y_3 + Y_1^2 \cos^2 Y_2 \cos^2 Y_3 \sin Y_3 + Y_1^2 \sin^2 Y_2 \sin^3 Y_3 + Y_1^2 \sin^2 Y_2 \cos^2 Y_3\right)\right|$$

$$= Y_1^2 \left| \sin Y_3 \cos^2 Y_2 (\sin^2 Y_3 + \cos^2 Y_3) + \sin^2 Y_2 (\cos^3 Y_3 + \sin^3 Y_3) \right|$$

$$+ \sin^2 Y_2 \cos^3 Y_3$$

$$= Y_1^2 \left| \sin Y_3 \cos^2 Y_3 + \sin^2 Y_2 \cos^3 Y_3 + \sin^2 Y_2 \sin^3 Y_3 \right|$$

$$= Y_1^2 \left|\right.$$

(4-47) $f(x) = \frac{1}{2}$; $-1 < x < 1$ $\quad Y = X^2$

$A_1 = (-1 < x < 0)$ $\quad x = -\sqrt{Y} \Rightarrow J_1 = \frac{-1}{2Y^{\frac{1}{2}}}$

$A_2 = (0 < x < 1)$ $\quad x = \sqrt{Y} \Rightarrow J_2 = \frac{1}{2Y^{\frac{1}{2}}}$

$B = (0 < Y < 1)$

$$Pr[Y \in B] = \int_{A_1} \frac{1}{2\sqrt{Y}} f[-\sqrt{Y}]\,dY + \int_{A_2} \frac{1}{2\sqrt{Y}} f[\sqrt{Y}]\,dY$$

$$= \int_B \frac{1}{2\sqrt{Y}} [f(-\sqrt{Y}) + f(\sqrt{Y})]$$

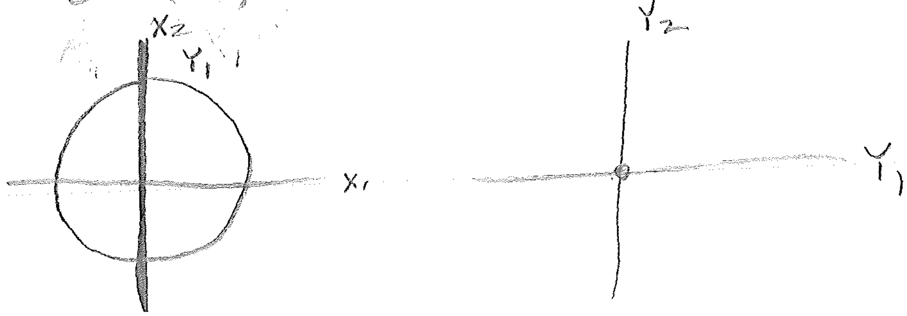
$$\Rightarrow f_Y(Y) = \frac{1}{2\sqrt{Y}} \left[\frac{1}{2} + \frac{1}{2}\right] = 1/2\sqrt{Y} \quad ; 0 < Y < 1$$

(4-48) $X_1, X_2 \sim n(0,1)$ $\quad -\infty < X_1 X_2 < \infty$

$Y_1 = X_1^2 + X_2^2 \qquad Y_2 = X_2$

$X_1 = \pm\sqrt{Y_2^2 - Y_1} \qquad X_2 = Y_2$

$B = \{ Y ; -\infty < Y < +\infty \}$



$A_1 = \{ (X_1, X_2) ; X_1 > 0 \}$

$A_2 = \{ (X_1, X_2) ; X_2 < 0 \}$

$$|J| = \begin{vmatrix} \mp \frac{1}{2\sqrt{Y_2^2 - Y_1}} & \pm Y_2/\sqrt{Y_2^2 - Y_1} \\ 0 & 1 \end{vmatrix}$$

$$\Rightarrow |J_1| = |J_2| = \frac{1}{2\sqrt{Y_2^2 - 1}}$$

$$\Rightarrow f(Y_1 Y_2) = \frac{1}{2\pi}\left[\int_{A_1} \frac{1}{2\sqrt{Y_2^2 - 1}} e^{-\frac{(Y_2^2 - Y_1) + Y_2^2}{2}}\right.$$

$$+ \int_{A_2} \frac{1}{2\sqrt{Y_2^2 - 1}} e^{-\frac{(Y_2^2 - 1) - Y_2^2}{2}}$$

$$= \int_{A_1 \cup A_2} \frac{1}{4\pi\sqrt{Y_2^2 - 1}} e^{-\frac{2Y_2^2 - Y_1}{2}}$$

(4-70) a. $Y = k_1X_1 + k_2X_2 + \cdots + k_nX_n$

$$= \sum_{i=1}^{n} k_i X_n$$

$$M_Y(t) = E\left[e^{tY}\right]$$

$$= E\left[e^{t\sum_{i=1}^{n} k_i X_n}\right]$$

$$= E\left[\prod_{i=1}^{n} e^{tk_iX_n}\right]$$

$$= \prod_{i=1}^{n} E\left[e^{tk_iX_n}\right] = \prod_{i=1}^{n} M(k_i t)$$

b. $M_X(t) = e^{m_i(e^t - 1)}$

$$\prod M_X(t) = e^{(e^t-1)\sum_{i=1}^{n} m_i}$$

$\Rightarrow$ POISSON

MEAN

$\mu = \sum m_i$

(4-71)

(a) $Y = \sum^n_i X_i$

$$M_Y(t) = E[e^{tY}]$$
$$= E[e^{t\sum X_i}]$$
$$= \prod_n^n E[e^{tX_i}]$$
$$= \prod M_i(t)$$
$$= M_i^n(t)$$

b. $Y = \frac{1}{n}\sum_{i=1}^n X_i$

$$M_Y = E[e^{t\frac{1}{n}\sum_{i=1}^n X_i}]$$
$$= \prod^n E[e^{t\frac{X_i}{n}}]$$
$$= M^n(\tfrac{X}{n})$$

(4-83) $X_i \quad i = 1, \ldots, 25 \quad n(0,16)$

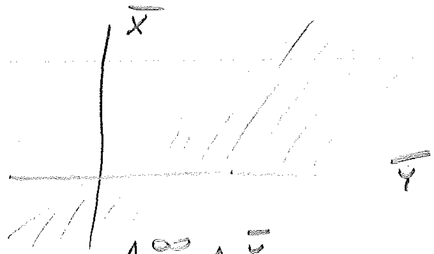
$Y_i \quad i = 1, \ldots, 25 \quad n(1,9)$

$\sigma = 20$

$\overline{X} \sim n(0,\ 25 \cdot 16) = n(0, 400)$

$\overline{Y} \sim n(25,\ 9 \cdot 25) = n(25, 225)$ $\quad \sigma = 15$

FIND $Pr[\overline{X} > \overline{Y}]$



$$= \int_{\overline{Y}=-\infty}^{\infty} \int_{\overline{X}=-\infty}^{\overline{Y}} n(25, 225)\, n(0, 400)\, d\overline{X}\, d\overline{Y}$$

(5·1) $\bar{X}_n$ = MEAN OF $n(\mu, \sigma^2)$

$$\bar{X}_n \sim n\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\Rightarrow \lim_{n \to \infty} \bar{X}_n = n(\mu, 0)$$

OR

$$F_{\bar{X}}(x) = \begin{cases} 0 ; & x < \mu \\ 1 ; & x \geq \mu \end{cases}$$

(5-4) $f_n(Y) = 1 \;;\; Y = n$

$$F_n(Y) = \begin{cases} 0 & ; \; Y < n \\ 1 & ; \; Y \geq n \end{cases}$$

$$\lim F_n(Y) = \begin{cases} 0 & ; \; Y < \infty \\ 1 & ; \; Y = \infty \end{cases}$$

WHICH IS GARBAGE

(5-6) $X_1, X_2, \cdots X_n \xrightarrow{s} n(\mu, \sigma^2)$

$$\Sigma X \sim n(n\mu, n\sigma^2)$$

$$\bar{X}_n \sim F_n = \frac{1}{\sqrt{2\pi}\sqrt{\frac{\sigma^2}{n}}} \int_{-\infty}^{x} e^{-\frac{(x-n\mu)^2}{\frac{2\sigma^2}{n}}} dx$$

$$x' = \frac{x - n\mu}{\sqrt{\sigma^2/n}}$$

$$\Rightarrow F_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x - n\mu}{\sqrt{\sigma^2/n}}} e^{-x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\sqrt{n}x - n^{\frac{3}{2}}\mu}{\sigma}} e^{-x^2/2} dx$$

$$\lim_{n\to\infty} F_n = \int_{-\infty}^{\infty} (\ ) dx = 0$$

(5-7) $Y_n \sim b(n, p)$

$\lim_{n \to \infty} Pr[|Y_n - p| < \epsilon]$

CHEBYCHEV SAYS

$Pr[|x - \mu| > k\sigma] \leq \frac{1}{k^2}$

BOB MARKS

(6-1) $f(x) = e^{-x} \quad ; 0 < x < \infty$

$$\begin{aligned}
P_r[X < 3 < 3X] &= P_r\left[1 < \tfrac{3}{X} < 3\right] \\
&= P_r\left[\tfrac{1}{3} < \tfrac{X}{3} < 1\right] \\
&= P_r[1 < X < 3] \\
&= \int_1^3 e^{-x}\,dx \\
&= -e^{-x}\Big|_1^3 \\
&= e^{-1} - e^{-3} = 0.318
\end{aligned}$$

$Y = 3X - X = 2X$

$E[Y] = 2E[X]$

$E[X] = 1 \Rightarrow E[Y] = 2$

(6-2) $f(x) = \frac{1}{4}$; $x = 1, 2, 3, 4$

$f(x_1, x_2) = \frac{1}{16}$

$Y = X_1 + X_2$

$$f(Y) = \begin{cases} \frac{1}{16} & Y = 2 \\ \frac{2}{16} & 3 \\ \frac{3}{16} & 4 \\ \frac{4}{16} & 5 \\ \frac{3}{16} & 6 \\ \frac{2}{16} & 7 \\ \frac{1}{16} & 8 \end{cases}$$

(OBTAINED FROM DISCRETE CONVOLUTION

$$Pr\left[1 < 3 < Y - \tfrac{1}{2}\right]$$
$$= Pr\left[Y - \tfrac{1}{2} > 3\right]$$
$$= Pr\left[Y > \tfrac{7}{2}\right]$$
$$= Pr\left[Y \geq 4\right]$$
$$= \tfrac{1}{16}(3 + 4 + 3 + 2 + 1)$$
$$= \tfrac{13}{16}$$

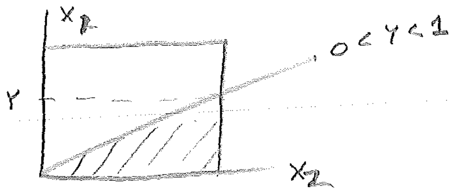
(6-3) $p = P_r\left[\frac{x_1}{3x_2} \le \frac{2}{3} \le \frac{2x_1}{x_2}\right]$

$\frac{x_1}{x_2} \le 2$ AND $\frac{x_1}{x_2} \ge \frac{1}{3} \Rightarrow \frac{1}{3} \le \frac{x_1}{x_2} \le 2$

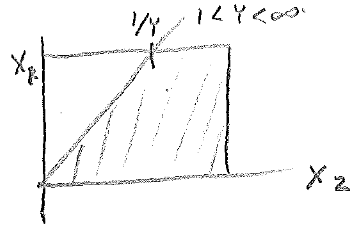
OR $\frac{x_2}{x_1} \le \frac{1}{2}$ AND $\frac{x_2}{x_1} \le 3 \Rightarrow$

LET $Y = \frac{x_1}{x_2} \Rightarrow x_1 = Y x_2$

$F(Y) = P_r[\mathrm{Y} \le Y] = P_r[X_1 \le Y X_2]$



$F(Y) = \frac{1}{2} Y \quad ; 0 < Y < 1$



$F(Y) = 1 - \frac{1}{2Y} \quad ; 1 < Y < \infty$

$$F(Y) = \begin{cases} \frac{1}{2} Y & ; 0 < Y < 1 \\ 1 - \frac{1}{2Y} & ; 1 < Y < \infty \end{cases}$$

$p = F(2) - F(\frac{1}{3})$

$= \left[1 - \frac{1}{4}\right] - \frac{1}{6}$

$= 1 - \frac{5}{12} = \frac{7}{12}$

(6-4) $\bar{X}$, n

FIND

$$p = Pr\left[\frac{n(\bar{X}-\mu)^2}{5.02} < \sigma^2 < \frac{n(\bar{X}-\mu)^2}{0.001}\right]$$

NOW $\bar{X} \sim n(\mu, \frac{\sigma^2}{n})$, $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim n(0,1)$

$$p = Pr\left[\frac{n}{5.02}\left(\frac{\bar{X}-\mu}{\sigma}\right)^2 \le 1 \le \frac{n}{0.001}\left(\frac{\bar{X}-\mu}{\sigma}\right)^2\right]$$

$$= Pr\left[\frac{1}{5.02}\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2 \le 1 \le \frac{1}{0.001}\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2\right]$$

$$\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2(1)$$

$$p = Pr\left[\frac{1}{5.02}\chi^2(1) \le 1 \le \frac{1}{0.001}\chi^2(1)\right]$$

$$= \cancel{Pr[\chi^2(1) \le 5.02] - Pr[\chi^2(1) \le 0.001]}$$

$$= Pr[\chi^2(1) \le 5.02,\ \chi^2(1) \ge 0.001]$$

$$= Pr[\chi^2(1) \le 5.02] - Pr[\chi^2(1) \le 0.001]$$

$$= 0.975 - 0.025$$

$$= 0.950$$

(b) $$L = \left[\frac{1}{0.001} - \frac{1}{5.02}\right]\sigma^2\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2$$

$$= 999.8\,\sigma^2\chi^2(1)$$

$$E[L] = 999.8\,\sigma^2 E[\chi^2(1)]$$

$\Rightarrow$ IN GENERAL, $E[\chi^2(r)] = r$

$$\Rightarrow E[L] = 999.8\,\sigma^2$$

THE RESULT HERE IS INDEPENDENT OF n.

IN EXAMPLE 4, OUR CONFIDENCE IS DEPENDENT ON n SINCE THE # OF DEGREES OF FREEDOM FOR THE STATISTIC Y IS n DEPENDENT.

which method is better?

(6-5) $p = P_r\left[\frac{S^2}{1.9} \le \sigma^2 < \frac{S^2}{0.27}\right]$

$$\frac{S^2}{1.9} \le \sigma^2 \quad \text{AND} \quad \frac{S^2}{0.27} > \sigma^2.$$

$$S^2 \le 1.9\sigma^2 \qquad S^2 > 0.25\sigma^2$$

$$p = P_r\left[0.25\sigma^2 \le S^2 < 1.9\sigma^2\right]$$

$$= P_r\left[0.25 \le \frac{S^2}{\sigma^2} \le 1.9\right]$$

$$= P_r\left[n\,0.27 \le \frac{nS^2}{\sigma^2} \sim \chi^2_{n-1} \le n\,1.9\right]$$

$$n = 10$$

$$\Rightarrow p = P_r\left[2.7 \le \chi^2_9 \le 19\right]$$

$$= 0.975 - 0.025 = 0.95$$

HERE, THE INTERVAL DOES DEPEND ON n ALSO, SINCE, AS IN EX. 4, THE DOF OF THE STATISTIC IS n DEPENDENT.

$$E[\text{LENGTH}] = E\left[\frac{S^2}{0.27} - \frac{S^2}{1.9}\right]$$

$$= \left[\frac{1}{0.27} - \frac{1}{1.9}\right]\frac{\sigma^2}{10} E\left(\frac{10S^2}{\sigma^2}\right)$$

$$= \left(\frac{1}{0.27} - \frac{1}{1.9}\right)\frac{\sigma^2}{10}\; 9$$

$$= 2.86\sigma^2$$

BOB MARKS
DUE: 1/27/77

(6.6) $\bar{X} = 81.2$ $n = 20$

$X_i \sim n(\mu, 80) \Rightarrow \bar{X} \sim n(\mu, \frac{80}{20}) = n(\mu, 4)$

FIND $b \ni$

$0.95 = P\left[-b \le \frac{\bar{X}-\mu}{2} \le b\right]$ (1)

$= P_r\left[-2b \le \bar{X}-\mu \le 2b\right]$

$= P_r\left[2b \ge \mu - \bar{X} \ge -2b\right]$

$= P\left[\bar{X} - 2b \le \mu < \bar{X} + 2b\right]$

CONF INT. IS $(\bar{X} - 2b, \bar{X} + 2b)$ (2)

FROM (1):

$P_r[-b \le z \le b] = 2P_r[0 \le z < b]$

$= 2\left[P_r[z < b] - P_r[z < 0]\right]$

$= 2P_r(z < b) - 1 = 0.95$

$\Rightarrow 2P_r(z < b) = 1.95$

$P_r(z < b) = 0.975$

$\Rightarrow b = 1.960$

FROM (2):

$(\bar{X} - 2b, \bar{X} + 2b)$

$= (81.2 - (1.96)2, 81.2 + (1.96)2)$

$= (77.28, 85.12)$

(6.7) $\bar{X}$, $n = ?$

$$X_i \sim n(\mu, 9)$$

$$\Rightarrow \bar{X} \sim n\left(\mu, \frac{9}{n}\right)$$

$$0.9 = P_r\left[\bar{X} - 1 < \mu < \bar{X} + 1\right]$$

$$= P_r\left[-1 < \mu - \bar{X} < 1\right]$$

$$= P_r\left[-1 < \bar{X} - \mu < 1\right]$$

$$= P_r\left[-\sqrt{\frac{n}{9}} < \frac{\bar{X} - \mu}{9/n} < \sqrt{\frac{n}{9}}\right]$$

$$= P_r\left[-\sqrt{\frac{n}{9}} < Z < \sqrt{\frac{n}{9}}\right]$$

$$= 2\, P_r\left[0 < Z < \sqrt{\frac{n}{9}}\right]$$

$$= 2\left[P_r\left(Z < \sqrt{\frac{n}{9}}\right) - P_r(Z < 0)\right]$$

$$= 2 P_r\left(Z < \sqrt{\frac{n}{9}}\right) - 1 = 0.9$$

$$2 P_r\left[Z < \sqrt{\frac{n}{9}}\right] = 1.9$$

$$P_r\left[Z < \sqrt{\frac{n}{9}}\right] = 0.95$$

$$\Rightarrow \sqrt{\frac{n}{9}} = 1.645$$

$$n = 9(1.645)^2 = 24.3$$

$\Rightarrow$ NEED @ LEAST 25 SAMPLES.

(6-8) $\frac{\bar{x}-\mu}{s/\sqrt{n-1}} \sim T_{n-1}$; $\bar{x} = 4.7$, $s^2 = 5.76$

$\Rightarrow P_r\left[-b \le \frac{\mu-\bar{x}}{s/\sqrt{n-1}} < b\right] = 0.9$

$= P_r[-b \le T_{16} < b]$

$= 2P_r[0 < T_{16} < b]$

$= 2P_r[T_{16} < b] - 1$

$\Rightarrow 2P_r[T_{16} < b] = 1.9$

$P_r[T_{16} < b] = 0.95$

$\Rightarrow b = 1.746$

$P_r\left[-b \le \frac{\mu-\bar{x}}{s/\sqrt{n-1}} < b\right]$

$\cancel{P_r[\bar{x} \cdot 4 \cdot b}$

$P_r\left[\bar{x} - \frac{bs}{\sqrt{n-1}} < \mu < \bar{x} + \frac{bs}{\sqrt{n-1}}\right]$

$\Rightarrow C.I. = \left[\bar{x} - \frac{bs}{\sqrt{n-1}} < \mu < \bar{x} + \frac{bs}{\sqrt{n-1}}\right]$

$= \left[4.7 - \frac{1.746 \cdot \sqrt{5.76}}{4}, 4.7 + \frac{1.746 \cdot \sqrt{5.76}}{4}\right.$

$= (3.7, 5.7)$

(6-9) $\bar{X}$, n, μ, $\sigma^2 = 10$

WE MUST ASSUME n IS SUFFICIENTY LARGE SO THAT THE CENTRAL LIMIT THEOREM (p. 182) MAY BE APPLIED

$$Y = \sqrt{n}\,(\bar{X} - \mu)/\sigma \sim n(0,1)$$
$$= \sqrt{n}\,(\bar{X} - \mu)/\sqrt{10}$$

NOW, FIND n SUCH THAT

$$0.954 = P_r(\bar{X} - \tfrac{1}{2} < \mu < \bar{X} + \tfrac{1}{2})$$
$$= P(-\tfrac{1}{2} < \mu - \bar{X} < \tfrac{1}{2})$$
$$= P(-\tfrac{1}{2} < \bar{X} - \mu < \tfrac{1}{2})$$

$$= P\left[-\frac{\sqrt{n}}{2\sqrt{10}} < \frac{\sqrt{n}(\bar{X} - \mu)}{\sqrt{10}} < \frac{\sqrt{n}}{2\sqrt{10}}\right]$$

$$= P_r\left[-\frac{1}{2}\sqrt{\frac{n}{10}} < z < \frac{1}{2}\sqrt{\frac{n}{10}}\right]$$

FROM EXAMPLE #3 IN THIS SECTION

$$\frac{1}{2}\sqrt{\frac{n}{10}} = 2$$
$$\sqrt{\frac{n}{10}} = 4$$
$$\frac{n}{10} = 16$$
$$n = 160$$

(6-10) $X_i \sim n(\mu, \sigma^2)$; $i = 1, 2, \ldots, 9$, $n = 9$

FIND 95% CONF. INTERVALS

(a) ASSUME σ IS KNOWN

$\Rightarrow Z = \dfrac{\sqrt{n}\,(\bar{X} - \mu)}{\sigma} = \dfrac{3(\bar{X} - \mu)}{\sigma} \sim n(0,1)$

$P_r[-2 < Z < 2] = 0.95 = P_r\left[-2 < \dfrac{3(\bar{X} - \mu)}{\sigma} < 2\right]$

$= P_r\left[\bar{X} - \dfrac{2\sigma}{3} < \mu < \bar{X} + \dfrac{2\sigma}{3}\right]$

$L_1 =$ LENGTH $= \dfrac{4\sigma}{3}$ ($\Rightarrow$ DETERMINISTIC)

(b) $(\sqrt{n-1})\dfrac{\bar{X} - \mu}{S} \sim t_{n-1}$

THIS FOLLOWS FROM

$Z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim n(0,1)$

$Y = nS^2/\sigma^2 \sim \chi^2_{n-1}$

$Z/\sqrt{Y/n-1} = \dfrac{\sqrt{n}(\bar{X} - \mu)/\cancel{\sigma}}{\sqrt{n}\, S/(\cancel{\sigma}\sqrt{n-1})} = \dfrac{(\bar{X} - \mu)(\sqrt{n-1})}{S} \sim t_{n-1}$

THUS, FOR $n = 9$

$0.95 = P_r[-b < t_8 < b] = 2\,P_r[0 < t_8 < b]$

$= 2P_r[t_8 < b] - 1 \Rightarrow P_r[t_8 < b] = \dfrac{1.95}{2} = 0.975$

$\Rightarrow b = 2.306$

$0.95 = P_r\left[-b < \sqrt{8}\,\dfrac{\bar{X} - \mu}{S} < b\right]$

$= P_r\left[\bar{X} - \dfrac{bS}{\sqrt{8}} < \mu < \bar{X} + \dfrac{bS}{\sqrt{8}}\right]$

$L_2 = \dfrac{2bS}{\sqrt{8}} = \dfrac{bS}{\sqrt{2}} = \dfrac{2.306}{\sqrt{2}}\, S = 1.631\, S$

(c) TO COMPARE L_1 AND L_2, LETS FIND $E[L_1^2]$ AND $E[L_2^2]$. NOW

$E[L_1^2] = \dfrac{16}{9}\sigma^2 = 1.778\sigma^2$

$E[L_2^2] = \dfrac{b^2}{2} E[S^2] = \dfrac{b^2}{2}\dfrac{\sigma^2}{n} E\left[\dfrac{nS^2}{\sigma^2}\right]$ ($\sim \chi^2_{n-1}$)

$= \dfrac{b^2}{2}\dfrac{\sigma^2}{n} E[\chi^2_{n-1}]$

$= \dfrac{b^2}{2}\dfrac{n-1}{n}\sigma^2 = \dfrac{(2.306)^2}{2}\dfrac{8}{9}\sigma^2$

$= 2.363\,\sigma^2$

AS SHOULD BE EXPECTED, $E[L_2^2] > E[L_1^2] = L_1^2$

SINCE, FOR L_1, WE HAVE MORE INFORMATION CONCERNING OUR SAMPLE STATISTIC.

(6-11)(a) $X_i \sim n(\mu, \sigma^2)$; $i = 1, 2, \ldots, n, n+1$

$$\bar{X} = \sum_{i=1}^{n} X_i/n \sim n\left(\mu, \frac{\sigma^2}{n}\right)$$

$$S^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n} \quad ; \quad \frac{nS^2}{\sigma^2} \sim \chi^2_{n-1}$$

FIND C SUCH THAT

$$\frac{C(\bar{X} - X_{n+1})}{S} \sim t$$

NOW $\quad \bar{X} - X_{n+1} \sim n\left(0, \left(\frac{1}{n} + 1\right)\sigma^2\right)$

$$\Rightarrow \frac{\bar{X} - X_{n+1}}{\sqrt{\frac{1}{n} + 1}\,\sigma} \sim n(0,1)$$

ALSO

$$\frac{nS^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$\Rightarrow \frac{(\bar{X} - X_{n+1})\,\cancel{\sigma}\sqrt{n-1}}{\sqrt{\frac{1}{n} + 1}\,\cancel{\sigma}\,\sqrt{n}\,S} \sim t_{n-1}$$

$$= \frac{(\bar{X} - X_{n+1})\sqrt{n-1}}{\sqrt{n+1}\;S} \Rightarrow C = \sqrt{\frac{n-1}{n+1}}$$

(b) $n = 8$, FIND $k \ni$

$$0.8 = P_r\left[\bar{X} - kS < X_9 < \bar{X} + kS\right]$$

$$= P_r\left[-k < \frac{X_9 - \bar{X}}{S} < k\right]$$

$$= P_r\left[-k\sqrt{\frac{n-1}{n+1}} < \sqrt{\frac{n-1}{n+1}}\left(\frac{\bar{X} - X_9}{S}\right) < k\sqrt{\frac{n-1}{n+1}}\right]$$

$$= P_r\left[-k\sqrt{\frac{n-1}{n+1}} < t_{n-1} < k\sqrt{\frac{n-1}{n+1}}\right]$$

NOW $\quad P_r\left[-b < t_{n-1} < b\right] = 0.8 = 2P_r\left[t < b\right] - 1$

$$\Rightarrow P_r\left[t_7 < b\right] = \frac{1.8}{2} = 0.9 \Rightarrow b = 1.415$$

THUS $\quad b = k\sqrt{\frac{n-1}{n+1}}$

$$\Rightarrow k = b\sqrt{\frac{n+1}{n-1}}$$

$$= (1.415)\sqrt{\frac{9}{7}}$$

$$= 1.60$$

(6-12) $Y \sim b(300, p)$; $n = 300$

SINCE n IS LARGE, USE CENTRAL LIMIT THEM:

$$\frac{Y - np}{\sqrt{np(1-p)}} \sim n(0,1)$$

NOW $P_r[-b < z < b] = 0.9 = 2P_r[z < b] - 1$

OR $P_r[z < b] = \frac{1.9}{2} = 0.95 \Rightarrow b = 1.645$

$P_r[-b < z < b] = P_r[0 < z^2 < b^2] = P_r[0 < \chi_1^2 < b^2]$

$$= P_r\left[0 \leq \frac{(Y-np)^2}{np(1-p)} < b^2\right]$$

$$= P_r[0 \leq Y^2 - 2npY + n^2p^2 < b^2np - b^2np^2]$$

$$= P_r[-b^2np(1-p) \leq p^2(n^2 + nb^2) - np(2Y + b^2) + Y^2 < 0]$$

THIS INEQ. IS ALWAYS SATISFIED FOR $0 < p < 1$

$$= P_r[np^2(n+b^2) - np(2Y+b^2) + Y^2 < 0]$$

$$= P_r[p^2\ 300(300 + 1.645^2) - p300(150 + 1.645^2) + 75^2 < 0]$$

$$= P_r[9.081 \times 10^4 p^2 - 4.581 \times 10^4 p + 5.625 \times 10^3 < 0]$$

USE QUADRADIC FORMULA ON p:

$$p = \frac{4.581 \times 10^4 \pm \sqrt{(4.581)^2 \times 10^8 - 4 \cdot 9.081 \times 5.625 \times 10^7}}{2 \cdot 9.081 \times 10^4}$$

$$= \frac{4.581 \times 10^4 \pm 7.438 \times 10^3}{18.162 \times 10^4}$$

$$= 0.293,\ 0.211$$

OR, IF ALL THE ARITHMETIC IS RIGHT:

$P_r[0.211 < p < .293] \approx 0.9$

(6-13) $\bar{X}$ $\quad X_i \sim n(\mu, \sigma^2)$ $\quad n, \sigma^2$ KNOWN

(a) FIND $c_1(\mu)$ AND $c_2(\mu)$ SUCH THAT

$$P_r[c_1(\mu) < \bar{X} < c_2(\mu)] = 0.954$$

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim n(0,1)$$

$$\Rightarrow P_r\left[-2 < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < 2\right] = 0.954 \quad (1)$$

$$= P_r\left[\mu - \frac{2\sigma}{\sqrt{n}} < \bar{X} < \mu + \frac{2\sigma}{\sqrt{n}}\right]$$

$$\Rightarrow c_1(\mu) = \mu - \frac{2\sigma}{\sqrt{n}}$$

$$c_2(\mu) = \mu + \frac{2\sigma}{\sqrt{n}}$$

(b) FIND $d_1(\bar{X})$ AND $d_2(\bar{X}) \ni$

$$P_r[d_2(\bar{X}) < \mu < d_1(\bar{X})]$$

FROM (1):

$$P_r\left[-\frac{2\sigma}{\sqrt{n}} < \bar{X} - \mu < \frac{2\sigma}{\sqrt{n}}\right] = 0.954$$

$$= P_r\left[\frac{-2\sigma}{\sqrt{n}} < \mu - \bar{X} < \frac{2\sigma}{\sqrt{n}}\right]$$

$$= P_r\left[\bar{X} - \frac{2\sigma}{\sqrt{n}} < \mu < \bar{X} + \frac{2\sigma}{\sqrt{n}}\right]$$

$$d_2(\bar{X}) = \bar{X} - 2\sigma/\sqrt{n}$$

$$d_1(\bar{X}) = \bar{X} + 2\sigma/\sqrt{n}$$

(SAME AS ON P. 194)

(6-14) $P_r[-2<Z<2]=0.954$

$= P_r[Z^2<4] = P_r[\chi_1^2<4]$

FOR LARGE n, WE USE CENTRAL LIMIT THEM:

$b(p,n) \cong n(np,\ np(1-p))$

IF $Y \sim b(p,n)$, THEN

$Z = \frac{Y-np}{\sqrt{np(1-p)}} \sim n(0,1)$ APPROXIMATELY

THUS

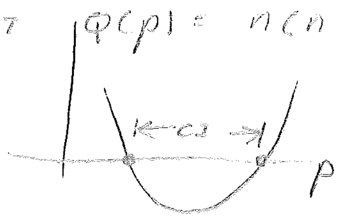
$P_r\left[\frac{(Y-np)^2}{np(1-p)} < 4\right] = 0.954$

$= P_r[Y^2 - 2nYp + n^2p^2 < 4np - 4np^2]$

$= P_r[(n^2+4n)p^2 - 2np(Y+2) + Y^2] < 0]$

$= P_r[n(n+4)p^2 - 2np(Y+2) + Y^2 < 0]$

LET $\varphi(p) = n(n+4)p^2 - 2np(Y+2) + Y^2$



THE ROOTS OF $\varphi(p)$ ARE @

$p = \frac{1}{\cancel{2}n(n+4)}\left[\cancel{2}n(Y+2) \pm \{\cancel{4}n^2(Y+2)^2 - \cancel{4}n(n+4)Y^2\}^{\frac{1}{2}}\right]$

$\frac{1}{n(n+4)}\left[(Y+2)n \pm \{n^2(Y+2)^2 - (n+4)Y^2 n\}^{\frac{1}{2}}\right]$

$\cancel{n^2Y^2} + 4n^2Y + 4n^2 - \cancel{n^2Y^2} - 4nY^2$

$4nY(n-Y) + 4n^2$

$\Rightarrow p = \frac{n(Y+2) \pm \{4nY(n-Y) + 4n^2\}^{\frac{1}{2}}}{n(n+4)}$

$= \frac{Y+2 \pm 2\left[\frac{Y(n-Y)}{n} + 1\right]^{\frac{1}{2}}}{(n+4)}$

(6-15) $\bar{X}$, $n = 25$, $\alpha = 4$, $\beta > 0$

$\mu = \alpha\beta$

$\sigma^2 = \alpha\beta^2$

$\bar{X} \sim n\left(\alpha\beta, \frac{\alpha\beta^2}{n}\right) = n\left(4\beta, \frac{4}{25}\beta^2\right)$

$$P_r\left[-2 < \frac{\bar{X} - \mu}{\frac{2}{5}\beta} < 2\right] = 0.954$$

BUT $\beta = \frac{\mu}{\alpha} = \frac{\mu}{4}$

$$\Rightarrow P_r\left[-2 < \frac{\bar{X} - \mu}{\frac{2}{5}\frac{\mu}{4}} < 2\right] = 0.954$$

$$= P_r\left[-\frac{4}{5}\frac{\mu}{4} < \bar{X} - \mu < \frac{4}{5}\frac{\mu}{4}\right]$$

$$= P_r\left[-\frac{\mu}{5} < \bar{X} - \mu < \frac{\mu}{5}\right]$$

$$= P_r\left[-\frac{1}{5} < \frac{\bar{X}}{\mu} - 1 < \frac{1}{5}\right]$$

$$= P_r\left[\frac{4}{5} < \frac{\bar{X}}{\mu} < \frac{6}{5}\right]$$

$$= P_r\left[\frac{5}{4} > \frac{\mu}{\bar{X}} > \frac{5}{6}\right]$$

$$= P_r\left[\frac{5}{6} < \frac{\mu}{\bar{X}} < \frac{5}{4}\right] \quad (\bar{X} > 0)$$

$$= P_r\left[\frac{5}{6}\bar{X} < \mu < \frac{5}{4}\bar{X}\right]$$

CONF. INTERVAL IS THEN

$$\left[\frac{5}{6}\bar{X}, \frac{5}{4}\bar{X}\right]$$

BOB MARKS
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(202)

10

(6-16) $n = m = 10$

$X_i \sim n(\mu_1, \sigma^2)$ $\qquad$ $Y_i \sim n(\mu_2, \sigma^2)$

$\bar{X} = 4.8$ $\qquad$ $S_1^2 = 8.64$

$\bar{Y} = 5.6$ $\qquad$ $S_2^2 = 7.88$

$0.95 = P_r[-b < t_{18} < b] = 2P_r[t_{18} \le b] - 1 \Rightarrow P_r[t_{18} \le b] = \frac{1.95}{2}$
$= 0.975$

$\Rightarrow b = 2.101$

$$b\sqrt{\frac{nS_1^2 + mS_2^2}{m+n-2}\left(\frac{1}{n} + \frac{1}{m}\right)}$$

$$= 2.101\sqrt{\frac{10(8.64 + 7.88)}{18}\left(\frac{1}{5}\right)}$$

$= 2.846$

$\bar{X} - \bar{Y} = -0.8$

$P_r[-0.8 - 2.846 < \mu_1 - \mu_2 < -0.8 + 2.846]$

$= P_r[-3.64 < \mu_1 - \mu_2 < 2.05]$

(6-17) $X_i \sim n(\mu_1, \sigma_1^2)$ $\qquad$ $Y_i \sim n(\mu_2, \sigma_2^2)$

$\bar{X} \sim n\left(\mu_1, \frac{\sigma_1^2}{n}\right)$ $\qquad$ $\bar{Y} \sim n\left(\mu_2, \frac{\sigma_2^2}{m}\right)$

$$\Rightarrow \bar{X} - \bar{Y} \sim n\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}\right)$$

$$\Rightarrow \frac{(\bar{X}-\bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}} \sim n(0,1)$$

σ_1^2 AND σ_2^2 ARE KNOWN. NOW, $\forall p \in (0,1)$

$\exists b \ni \Pr[-b < z < b] = p$

THUS

$$p = \Pr\left[-b < \frac{(\mu_1 - \mu_2) - (\bar{X} - \bar{Y})}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}} < b\right]$$

$$= \Pr\left[(\bar{X} - \bar{Y}) - b\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}} < \mu_1 - \mu_2 < (\bar{X} - \bar{Y}) + b\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}\right]$$

NO PROBLEM. OUR CONFIDENCE INTERVAL IS

$$\left[(\bar{X} - \bar{Y}) - b\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}\,;\ (\bar{X} - \bar{Y}) + b\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}\right]$$

(6-15) CONSIDER $\quad X_i \sim n(\mu_1, \sigma_1^2) \qquad Y_i \sim n(\mu_2, \sigma_2^2)$

$\bar{X} \sim n\left(\mu_1, \frac{\sigma_1^2}{n}\right) \quad ; \quad \bar{X}_2 \sim n\left(\mu_2, \frac{\sigma_2^2}{m}\right)$

$\Rightarrow$ n SAMPLES FROM X & m FROM Y.

NOW $\quad \bar{X} - \bar{Y} \sim n\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}\right)$

CONSIDER THE SAMPLE VARIANCES:

$$\frac{nS_1^2}{\sigma_1^2} \sim \chi^2_{n-1} \qquad \frac{mS_2^2}{\sigma_2^2} \sim \chi^2_{m-1}$$

$$\Rightarrow \quad \frac{nS_1^2}{\sigma_1^2} + \frac{mS_2^2}{\sigma_2^2} \sim \chi^2_{n+m-2}$$

PROCEEDING IN THE FASHION AS BEFORE

$$\frac{\left[(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)\right] \Big/ \sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}}{\sqrt{\left(\frac{nS_1^2}{\sigma_1^2} + \frac{mS_2^2}{\sigma_2^2}\right) \Big/ (m+n-2)}} \sim t_{m+m-2}$$

BUT, FOR $\sigma_1^2 \neq \sigma_2^2$, WE ARE LEFT WITH UNKNOWN σ^2 TERMS IN OUR CONFIDENCE INTERVAL. (WHEN $\sigma_1 = \sigma_2$, THEY CANCEL OUT). REARRANGING:

$$\frac{\left[\bar{X} - \bar{Y} - (\mu_1 - \mu_2)\right] \sqrt{m+n-2}}{\left[\left(\frac{nS_1^2}{\sigma_1^2} + \frac{mS_2^2}{\sigma_2^2}\right)\left(\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}\right)\right]^{\frac{1}{2}}}$$

$$= \frac{\left[\bar{X} - \bar{Y} - (\mu_1 - \mu_2)\right] \sqrt{m+n-2}}{\left[\left(nS_1^2 + mS_2^2 \frac{\sigma_1^2}{\sigma_2^2}\right)\left(\frac{1}{n} + \frac{1}{m}\frac{\sigma_2^2}{\sigma_1^2}\right)\right]^{\frac{1}{2}}}$$

IF WE KNOW THE RATIO OF VARIANCES SAY $K = \sigma_1^2 / \sigma_2^2$, THEN

$$T_{n+m-2} = \frac{\left[\bar{X} - \bar{Y} - (\mu_1 - \mu_2)\right] \sqrt{n+m-2}}{\left[(nS_1^2 + mS_2^2 K)\left(\frac{1}{n} + \frac{1}{mK}\right)\right]^{\frac{1}{2}}} \sim t_{n+m-2}$$

NOW WE CAN MAKE ~~A CAN~~ MAKE A CONFI. INTERVAL WITH ALL KNOWN PARAMETERS. $\exists\, b \ni$

$$q = P_r\left[-b < t_{n+m-2} < b\right]$$

OUR q^{TH} PERCENT CONFIDENCE INTERVAL IS THEN GIVEN BY $\quad q = P_r\left[(\bar{X} - \bar{Y}) - R < \mu_1 - \mu_2 < (\bar{X} - \bar{Y}) + R\right]$

WHERE

$$R = \frac{b\left[(nS_1^2 + mS_2^2 K)\left(\frac{1}{n} + \frac{1}{mK}\right)\right]^{\frac{1}{2}}}{\sqrt{n+m-2}}$$

(6-19) GENERALLY

$$P_r\left[(\bar{X}-\bar{Y}) - b\sigma\sqrt{\tfrac{1}{n}+\tfrac{1}{m}} < \mu_1 - \mu_2 < (\bar{X}-\bar{Y}) + b\sigma\sqrt{\tfrac{1}{n}+\tfrac{1}{m}}\right] = p$$

WHERE

$$p = P_r[-b < Z < b]$$

$$p = 0.9 = 2P_r[Z < b] - 1 \Rightarrow P_r[Z<b] = \frac{1.9}{2} = 0.95$$

$$\Rightarrow b = 1.645$$

SET

$$b\cancel{\sigma}\sqrt{\frac{1}{n}+\frac{1}{m}} = \frac{1}{5}\cancel{\sigma}$$

SINCE $n = m$

$$b\sqrt{\frac{2}{m}} = \frac{1}{5}$$

$$\sqrt{\frac{2}{m}} = \frac{1}{5b}$$

$$\frac{m}{2} = 25b^2$$

$$m = 50b^2 = 50(1.645)^2 = 135.3$$

$$\Rightarrow n = 135 \text{ OR } 136$$

(6-20) $$\frac{\frac{Y_1}{n_1}\left(1-\frac{Y_1}{n_1}\right)\frac{1}{n_1}+\frac{Y_2}{n_2}\left(1-\frac{Y_2}{n_2}\right)\frac{1}{n_2}}{P_1(1-P_1)/n_1+P_2(1-P_2)/n_2} \qquad (1)$$

CONSIDER FIRST

$$Y_i \sim b(n_i, p_i) \overset{\text{APPROX}}{\sim} Z(n_i p_i,\ n_i p_i(1-p_i))$$

$$\frac{Y_i}{n_i} \sim Z\left(p_i,\ \frac{p_i(1-p_i)}{n_i}\right) \qquad ; i = 1,2$$

WE WISH TO SHOW (VIA THEM 1 ON p. 176)

$$\lim_{n_i\to\infty} P_r\left[\left|\frac{Y_i}{n_i}-p_i\right|<\epsilon\right]=1 \qquad (2)$$

BY CHEBYCHEV'S INEQUALITY (THEM 7, p. 55)

$$P_r\left[\left|\frac{Y_i}{n_i}-p\right|\ge c\right]\le\frac{\sigma^2}{c^2}=\frac{p_i(1-p_i)}{n_i c^2}$$

THUS

$$\lim_{n_i\to\infty} P_r\left[\left|\frac{Y_i}{n_i}-p\right|\ge c\right]=0$$

AND PROPOSITION (2) IS PROVED. THUS, Y_i/n_i CONVERGES STOCHASTICALLY TO p_i

USING THE RESULTS ON p. 189 (PROB 5.35), WE CAN SAY THAT (1) THEN CONVERGES STOCHASTICALLY TO

$$\frac{P_1(1-P_1)\frac{1}{n_1}+P_2(1-P_2)\frac{1}{n_2}}{P_1(1-P_1)\frac{1}{n_1}+P_2(1-P_2)\frac{1}{n_2}} = 1$$

BOB MARKS
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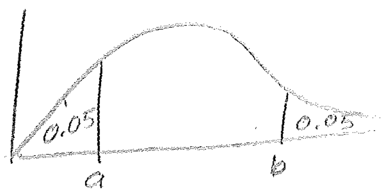
(6-21) $x_i = 8.6, 7.9, 8.3, 6.4, 8.4, 9.8, 7.2, 7.8, 7.5$

$\mu = 8 \Rightarrow |x_i - \mu| = 0.6, 0.1, 0.3, 1.6, 0.4, 1.8, 0.8, 0.2, 0.5$

$\Sigma |x_i - \mu|^2 = 7.35 \quad \sum \frac{(x_i - \mu)^2}{\sigma^2} \sim \chi^2_9$

$n = 9$



$a = 3.33 \qquad b = 16.9$

CONF INT. IS

$\left[\frac{7.35}{16.9}, \frac{7.35}{3.33}\right] = [0.435, 2.21]$

(6-22)

$$\left[\frac{\Sigma(x_i-\mu)^2}{b} , \frac{\Sigma(x_i-\mu)^2}{a} \right]$$

$$L = \Sigma(x_i-\mu)^2 \left[\frac{1}{a} - \frac{1}{b} \right] > 0$$

$$= \Sigma(x_i-\mu) \quad \frac{b-a}{ab}$$

$$E[L] = \frac{b-a}{ab} \; E\left[\frac{\Sigma(x_i-\mu)^2}{\sigma^2} \right] \sigma^2$$

NOW $\therefore \frac{\Sigma(x_i-\mu)^2}{\sigma^2} \sim \chi^2(n)$

$$\Rightarrow E\left[\frac{\sum^n (x_i-\mu)^2}{\sigma^2} \right] = n$$

AND

$$E[L] = \frac{(b-a)n\sigma^2}{ba}$$

(6.23) $n = 15$

$\bar{x} = 3.2$

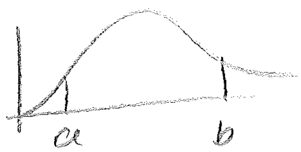
$s^2 = 4.24$

$n(\mu, \sigma^2)$

ASSUME μ & σ^2 <u>NOT</u> KNOWN

$$\frac{ns^2}{\sigma^2} \sim \chi^2_{n-1} = \chi^2_{14}$$

90%



$a = 6.57, \quad b = 23.7$

$$0.9 = P_r\left[a < \frac{ns^2}{\sigma^2} < b\right]$$

$$= P_r\left[\frac{1}{b} < \frac{\sigma^2}{ns^2} < \frac{1}{a}\right]$$

$$= P_r\left[\frac{ns^2}{b} < \sigma^2 < \frac{ns^2}{a}\right]$$

$$= P_r\left[\frac{15 \cdot 4.24}{23.7} < \sigma^2 < \frac{15 \cdot 4.24}{6.57}\right]$$

$$= P_r\left[2.7 < \sigma^2 < 9.7\right]$$

CONF. INT. IS

$[2.7, 9.7]$

(6.24) $n = 16$ $m = 10$

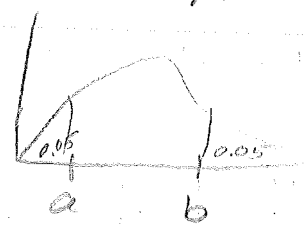
$n(\mu_1, \sigma_1^2)$ $n(\mu_2 \sigma_2^2)$

$\overline{X} = 3.6$ $\overline{Y} = 13.6$

$S_1^2 = 4.14$ $S_2^2 = 7.26$

NOW, FIND

$$P_r[a < F_{15,9} < b]$$



$b = 3.01$

$$F_{15,9} = \frac{1}{F_{9,15}}$$

$$\Rightarrow a = \frac{1}{2.59}$$

NOW $$\frac{m S_2^2 (n-1)}{n S_1^2 (m-1)} = \frac{10 \cdot 7.26 \cdot 15}{16 \cdot 4.14 \cdot 9}$$

$$= 1.826$$

90% C.I. IS THEN

$$[a \cdot 1.826, b \cdot 1.826]$$

$$= \left[\frac{1.826}{2.59}, 3.01 \cdot 1.826\right]$$

$$= [0.71, 5.50]$$

(6.25)

LET $U = \dfrac{\sum^{n} (X_i - \mu_1)^2}{\sigma_1^2} \sim \chi^2(n)$

$V = \dfrac{\sum^{m} (Y_i - \mu_2)^2}{\sigma_2^2} \sim \chi^2(m)$

THEN

$$\frac{U/n}{V/m} \sim F_{n,m}$$

AND, CHOOSE a & b GIVEN p ∋

$$p = Pr[a < F_{n,m} < b]$$

THEN

$$p = Pr\left[a < \frac{U/n}{V/m} < b\right]$$

$$= Pr\left[a < \frac{\sum (X_i - \mu_1)^2}{\sigma_1^2 n} \frac{m\sigma_2^2}{\sum (Y_i - \mu_2)^2} < b\right]$$

THEN

$$p = Pr\left[a \frac{n}{m} \frac{\sum (Y_i - \mu_2)^2}{\sum (X_i - \mu_1)^2} < \frac{\sigma_2^2}{\sigma_1^2} < b \frac{n}{m} \frac{\sum (Y_i - \mu_2)^2}{\sum (X_i - \mu_1)^2}\right]$$

(NO PROBLEM)

(6-26) THE GAMMA MGF IS

$$M(t) = \frac{1}{(1-\beta t)^{\alpha}} \quad ; t < \frac{1}{\beta}$$

THUS, $Y = \sum^{6} X_i$ HAS MGF

$$M(t) = \frac{1}{(1-\beta t)^{6\alpha}}$$

$\Rightarrow$ IF $X \sim \Gamma(\alpha, \beta)$

THEN $\Rightarrow$ $Y \sim \Gamma(6\alpha, \beta)$

GIVEN $\alpha = 1$, WE HAVE

$$Y \sim \Gamma(6, \beta)$$

IN GENERAL

$$f_{\Gamma}(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} X^{\alpha-1} e^{-x/\beta}$$

FOR $\alpha = 1$

$$f_{\Gamma}(x) = \frac{1}{\Gamma(1)\beta} e^{-x/\beta}$$

NOW

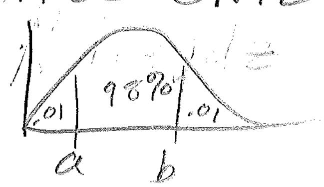
$$f_Y(Y) = \frac{1}{\Gamma(6)\beta^6} Y^5 e^{-Y/\beta}$$

LET $Z = \frac{2Y}{\beta} \Rightarrow Y = \frac{\beta}{2} Z$

$$f(Z) = \frac{\cancel{\beta}}{2} \frac{1}{\Gamma(6)\cancel{\beta^6}} \left(\frac{\cancel{\beta} Z}{2}\right)^5 e^{-\cancel{\beta} Z/2\cancel{\beta}}$$

$$= \frac{1}{2\Gamma(6)} \frac{Z^5}{2^5} e^{-Z/2} = \frac{Z^5}{\Gamma(6)2^6} e^{-Z/2}$$

NOTE, $f(Z)$ IS IND. OF β. $f(Z)$ IS RECOGNIZED AS χ^2_{12}.



$a = 3.57$

$b = 26.2$

$$0.98 = P_r[a < \chi^2_{12} < b]$$

$$= P_r\left[a < \frac{2Y}{\beta} < b\right]$$

$$= P_r\left[\frac{1}{b} < \frac{\beta}{2Y} < \frac{1}{a}\right]$$

$$= P_r\left[\frac{2}{b}\sum_{i=1}^{6} X_i < \beta < \frac{2}{a}\sum_{i=1}^{6} X_i\right]$$

$$= P_r\left[0.763 \sum^{6} X_i < \beta < 0.560 \sum^{6} X_i\right]$$

(6.27) $\dfrac{nS_1^2 + mS_2^2}{\sigma^2} \sim \chi^2(n+m-2)$

APPROPRIATELY FIND a & b $\ni$

$$p = P_r\left[a < \chi^2_{n+m-2} < b\right]$$

THEN

$$p = P_r\left[a < \frac{nS_1^2 + mS_2^2}{\sigma^2} < b\right]$$

$$= P_r\left[\frac{1}{b} < \frac{\sigma^2}{nS_1^2 + mS_2^2} < \frac{1}{a}\right]$$

$$= P_r\left[\frac{nS_1^2 + mS_2^2}{b} < \sigma^2 < \frac{nS_1^2 + mS_2^2}{a}\right]$$

$100p^{TH}$ CONFIDENCE INTERVAL IS :-

$$\left[\frac{nS_1^2 + mS_2^2}{b}, \frac{nS_1^2 + mS_2^2}{a}\right]$$

(6.29) $X_i \sim b(1,\theta)$, $Y = \sum_{i=1}^{n} X_i$

FROM EXAMPLE 2 ON p. 185

$Y \sim b(n,p)$

$\Rightarrow g(Y|\theta) = \binom{n}{Y}\theta^{Y}(1-\theta)^{n-Y}$; $Y = 0,\dots,n$

FROM 4.25 ON p. 134

$$h(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} ;\ 0<\theta<1 \quad (1)$$

(a) $k(\theta,Y) = g(Y|\theta)h(\theta)$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\binom{n}{Y}\theta^{\alpha+Y-1}(1-\theta)^{n+\beta-Y-1}$$

; $Y = 0,\dots,n$, $0<\theta<1$

(b) $k_1(Y) = \int_0^1 k(\theta,Y)\,d\theta$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\binom{n}{Y}\int_0^1 \theta^{\alpha+Y-1}(1-\theta)^{n+\beta-Y-1}\,d\theta$$

FROM 4.26 ON p. 134:

$$\int_0^1 Y^{\alpha-1}(1-Y)^{\beta-1}\,dY = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (2)$$

THUS

$$k_1(Y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\binom{n}{Y}\frac{\Gamma(\alpha+Y)\Gamma(n+\beta-Y)}{\Gamma(\alpha+n+\beta)}$$

(c) $k(\theta/Y) = k(\theta,Y)/k_1(Y)$

$$= \cancel{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}\cancel{\binom{n}{Y}}\cancel{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}}\cancel{\frac{1}{\binom{n}{Y}}}$$

$$\times \frac{\Gamma(\alpha+Y)\Gamma(n+\beta-Y)}{\Gamma(\alpha+n+\beta)}\theta^{\alpha+Y-1}(1-\theta)^{n+\beta-Y-1}$$

$$\cancel{E[k(\theta/Y)] = \int_0^1 \theta\, k(\theta/Y)\,d\theta}$$

$$\cancel{= \frac{\Gamma(\alpha+Y)\Gamma(n+\beta-Y)}{\Gamma(\alpha+n+\beta)}\int_0^1 \theta^{(\alpha+Y+1)-1}(1-\theta)^{(n+\beta-Y)-1}\,d\theta}$$

~~FROM (1):~~

$$\cancel{E[k(\theta/Y)] = \frac{\Gamma(\alpha+Y)\Gamma(n+\beta-Y)}{\Gamma(\alpha+n+\beta)}\frac{\Gamma(\alpha+Y+1)\Gamma(n+\beta-Y)}{\Gamma(\alpha+n+\beta+1)}}$$

$$k(\theta/Y) = \frac{\Gamma(\alpha+Y)\Gamma(n+\beta-Y)}{\Gamma(\alpha+n+\beta)}\theta^{\alpha+Y-1}(1-\theta)^{n+\beta-Y-1}$$

THIS IS A β DISTRIBUTION WITH

$\alpha_1 = \alpha + Y$ $\quad\quad$ $\beta_1 = n+\beta-Y$

(CONT) →

FROM 4.2C, P.134, WE SHOWED THAT, IF

$$Z \sim \mathcal{B}(\alpha', \beta') = \frac{\Gamma(\alpha'+\beta')}{\Gamma(\alpha')\Gamma(\beta')} Z^{\alpha'-1}(1-Z)^{\beta'-1}$$

THEN $\mu_Z = \frac{\alpha'}{\alpha'+\beta'}$ $\quad \sigma_Z^2 = \frac{\alpha'\beta'}{(\alpha'+\beta'+1)(\alpha'+\beta')}$

HERE: $\alpha' = \alpha + Y$ $\quad \beta' = n + \beta - Y$

$$\Rightarrow E[Y] = \frac{\alpha+Y}{\alpha+n+\beta}$$

$$Var(Y) = \frac{(\alpha+Y)(n+\beta-Y)}{(\alpha+n+\beta+1)(\alpha+n+\beta)^2}$$

(d) $k(\theta/Y) = \mathcal{B}_Z(\alpha', \beta') = \frac{\Gamma(\alpha'+\beta')}{\Gamma(\alpha')\Gamma(\beta')} Z^{\alpha'-1}(1-Z)^{\beta'-1}$

ASSUME WE HAD TABLES OF

$$P_r[Z \le z] = \begin{cases} 0 & ; z < 0 \\ \int_0^z \mathcal{B}_Z(\alpha';\beta')dz & ; 0 < z < 1 \\ 1 & ; z > 1 \end{cases}$$

THEN, FIND AN a AND b $\ni$

$$P_r[0 \le Z < a] = \frac{1-\alpha}{2}$$

$$P_r[b \le Z < 1] = \frac{1-\alpha}{2}$$

$$\Longrightarrow P_r[a \le Z < b] = \alpha$$

THEN, THE BAYSIAN CONFIDENCE INTERVAL FOR θ CONDITIONED ON $Y = y$ IS

$$[a, b]$$

WHERE $\alpha' = \alpha + Y$

AND $\beta' = n + \beta - Y$

(6-32) $X \sim n(0, \frac{1}{\theta})$

$\theta \sim \Gamma(\frac{r}{2}, \frac{2}{r})$ WHERE, FROM p. 99:

$$\Gamma(\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}$$

$$\Rightarrow \theta \sim \frac{1}{\Gamma(\frac{r}{2})(\frac{2}{r})^{r/2}} x^{\frac{r}{2}-1} e^{-xr/2}$$

$$g(x/\theta) = \frac{\sqrt{\theta}}{\sqrt{2\pi}} e^{-\frac{x^2\theta}{2}} \checkmark$$

$$k(x,\theta) = g(x|\theta)\,h(\theta)$$

$$= \sqrt{\frac{\theta}{2\pi}} e^{-\frac{\theta}{2}x^2} \frac{1}{\Gamma(\frac{r}{2})(\frac{2}{r})^{r/2}} \checkmark$$

$$\theta^{\frac{r}{2}-1} e^{-\theta r/2}$$

$$k_1(x) = \frac{1}{\Gamma(\frac{r}{2})(\frac{2}{r})^{r/2}} \int_0^\infty \sqrt{\frac{\theta}{2\pi}} e^{-\frac{\theta}{2}(r+x^2)} \theta^{\frac{r}{2}-1} d\theta$$

$$= \left(\frac{1}{\Gamma(\frac{r}{2})(\frac{2}{r})^{r/2}\sqrt{2\pi}}\right) \int_0^\infty \theta^{\frac{r-1}{2}} e^{-\frac{\theta}{2}(r+x^2)} d\theta$$

LET $\theta' = \frac{\theta}{2}(r+x^2)$

$$\Rightarrow \theta = \frac{2\theta'}{r+x^2} \Rightarrow d\theta = \frac{2\,d\theta'}{r+x^2}$$

$$k_1(x) = (\quad) \frac{2}{r+x^2} \int_0^\infty \left(\frac{2\theta'}{r+x^2}\right)^{\frac{r-1}{2}} e^{-\theta'} d\theta$$

$$= (\quad) \left(\frac{2}{r+x^2}\right)^{\frac{r+1}{2}} \int_0^\infty \theta'^{(\frac{r+1}{2})-1} e^{-\theta} d\theta$$

$$= (\quad) \left(\frac{2}{r+x^2}\right)^{\frac{r+1}{2}} \Gamma\left(\frac{r+1}{2}\right) \quad \binom{\text{FROM}}{\text{p. 99}}$$

$$= \frac{\left(\frac{2}{r+x^2}\right)^{\frac{r+1}{2}} \Gamma(\frac{r+1}{2})}{\sqrt{2\pi}\,\Gamma(\frac{r}{2})\,(2/r)^{r/2}} = \frac{\Gamma(\frac{r+1}{2})}{\sqrt{2\pi r}\,\Gamma(\frac{r}{2})} \frac{\left(\frac{2}{r+x^2}\right)^{\frac{r+1}{2}}}{\left(\frac{2}{r}\right)^{\frac{r+1}{2}}}$$

$$= \frac{\Gamma(\frac{r+1}{2})}{\sqrt{2\pi r}\,\Gamma(\frac{r}{2})\left(1+\frac{x^2}{r}\right)^{\frac{r+1}{2}}} \checkmark$$

THIS IS t DIST WITH r D.O.F. (p. 401 TOP)

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(7-1) $X_i \sim f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$; $0 < x, \theta < \infty$

$\bar{X} = \frac{1}{n} \sum^{n} X_i$

BY EXAMPLE 2 ON p.167

$E[\sum_n X_i] = n \, E[X] = n\theta$

$\Rightarrow E[\bar{X}] = \theta$

NOW $var(X) = \theta^2$

$\Rightarrow var(\sum_n X_i) = n \, \theta^2$

THUS

$var(\bar{X}) = var\left(\frac{\sum X_i}{n}\right) = \left(\frac{1}{n}\right)^2 n\theta^2$

$= \theta^2/n$

(7-2) $X_i \sim n(0,\theta) \qquad ; 0<\theta<\infty$

$$Y = \sum^{n} X_i^2 / n$$

$$E[Y] = \frac{1}{n} E\left[\sum_n X_i^2\right]$$

$$\frac{X_i^2}{\theta} \sim \chi_1^2$$

$$\Rightarrow \sum_n \frac{X_i^2}{\theta} \sim \chi_n^2$$

$$E[Y] = \frac{\theta}{n} E\left[\frac{\sum_n X_i^2}{\theta}\right]$$

$$= \frac{\theta}{n} n = \theta$$

$$Var\, Y = \frac{1}{n^2} Var \sum_n X_i^2$$

$$= \frac{\theta^2}{n^2} Var \frac{\sum_n X_i^2}{\theta}$$

$$var\, \chi_n^2 = 2n$$

$$\Rightarrow Var\, Y = \frac{\theta^2 (2\cancel{n})}{n^2} = \frac{2\theta^2}{n}$$

(7-4) $2 \operatorname{Var} Y_2 = \operatorname{var} Y_1$

$E[Y_1] = \theta = E[Y_2]$

$Z = K_1 Y_1 + K_2 Y_2$

$E[k_1 Y_1 + k_2 Y_2] = (k_1 + k_2)\theta$

$\operatorname{Var} Z = k_1^2 \operatorname{var} Y_1 + k_2^2 \operatorname{var} Y_2$

$= (2k_1^2 + k_2^2) \operatorname{var} Y_1 \quad (1)$

IF Z IS TO BE AN UNBIASED STATISTIC OF θ, THEN WE REQUIRE

$k_1 + k_2 = 1 \Rightarrow k_2 = 1 - k_1$

(1) BECOMES

$\operatorname{Var} Z = [2k_1^2 + (1-k_1)^2] \operatorname{var} Y_1$

$= [2k_1^2 + k_1^2 - 2k_1 + 1] \operatorname{var} Y_1$

$\frac{d}{dk_1}(3k_1^2 - 2k_1 + 1) = 6k_1 - 2 = 0 = 3k_1 - 1$

$\Rightarrow k_1 = \frac{1}{3}$

$k_2 = \frac{2}{3}$

$$f(\theta - x) = f(\theta + x) \quad \forall\, x$$

(7-5) $X_i \sim f(x)$

$$U(x_1 + h, \dots, x_n + h) = U(x_1 \dots x_n) + h \quad \forall\, h$$

$$U(-x_1, \dots, -x_n) = -U(x_1 \dots x_n)$$

$$E[U(x_1 \dots x_n) - \theta] = \int_{x_1} \cdots \int_{x_n} [U(x_1 \dots x_n) - \theta]\, f(x_1) \cdots f(x_n)\, dx_1 \dots dx$$

$$x_i = y_i + \theta$$

$$E[U(x_1 \dots x_n) - \theta] = \int_{y_1} \cdots \int_{y_n} [U(y_1 + \theta, \dots, y_n + \theta) - \theta]\, f(y_1 + \theta) \cdots f(y_n + \theta)\, dy_1 \dots dy_n$$

$$= \int_{y_1} \cdots \int_{y_n} U(y_1 \dots y_n)\, f(y_1 + \theta) \cdots f(y_n + \theta)\, dy_1 \dots dy_n$$

$$= K$$

$$z_i = -y_i$$

$$K = \int_{z_1} \cdots \int_{z_n} U(-z_1, \dots -z_n)\, f(\theta - z_1) \cdots f(\theta - z_n)\, dz_1 \cdots dz_n$$

$$= -\int_{z_1} \cdots \int_{z_n} U(z_1 \dots z_n)\, f(\theta + z_1) \cdots f(\theta + z)\, dz_1 \cdots dz_n$$

$$= -K$$

THUS $K = -K = 0$ AND $U(x_1 \dots x_n)$ IS AN UNBIASED STATISTIC OF θ

(7-6) $X_K \sim n(0,\theta) \Rightarrow \frac{X_K}{\sqrt{\theta}} \sim n(0,1)$

$$\frac{X_K^2}{\theta} \sim \chi_1^2$$

$$\frac{1}{\theta}\sum_{K=1}^{n} X_K^2 \sim \chi_n^2 = \frac{1}{\Gamma(\frac{n}{2})\,2^{n/2}}\, x^{\frac{n}{2}-1} e^{-x/2}$$

$$\Rightarrow \sum_{K=1}^{n} X_K^2 \sim \frac{\frac{1}{\theta}}{\Gamma(\frac{n}{2})\,2^{n/2}} \left(\frac{x}{\theta}\right)^{\frac{n}{2}-1} e^{-x/2\theta}$$

$$\prod_{K=1}^{n} f(x_K;\theta) = \left(\frac{1}{\sqrt{2\pi\theta}}\right)^n \prod_{K=1}^{n} e^{-\frac{x_K^2}{2\theta}}$$

$$= \left(\frac{1}{\sqrt{2\pi\theta}}\right)^n e^{-\Sigma x_K^2/2\theta}$$

$$= \frac{1}{\sqrt{2\pi}}\,\theta^{-\frac{n}{2}} e^{-\Sigma x_K^2/2\theta} = g[\Sigma x_K^2;\theta]$$

$$\sum_{K=1}^{n} x_K^2 \sim \frac{1}{\Gamma(\frac{n}{2})\,2^{n/2}} (\Sigma x^2)^{\frac{n}{2}-1}\, \frac{\theta}{\theta}^{1-\frac{n}{2}} e^{-\Sigma x_K^2/2\theta}$$

$$= \frac{1}{\Gamma(\frac{n}{2})\,2^{n/2}} (\Sigma x^2)^{\frac{n}{2}-1}\, \theta^{-\frac{n}{2}} e^{-\Sigma x_K^2/2\theta}$$

$$= \prod_{K=1}^{n} f(x_K;\theta)\,\sqrt{2\pi}\,\frac{1}{\Gamma(\frac{n}{2})\,2^{-n/2}} (\Sigma x^2)^{\frac{n}{2}-1}$$

$$\Rightarrow \prod_{K=1}^{n} f(x_K;\theta)$$

$$= g[\Sigma x_K^2;\theta]\,\underbrace{\frac{\Gamma(\frac{n}{2})\,2^{-\frac{n}{2}}}{\sqrt{2\pi}}\,(\Sigma x^2)^{1-\frac{n}{2}}}_{H}$$

THUS, BY THEM. 1 ON 216,
Σx^2 IS A SUFFICIENT STATISTIC

(7-7) $X_i \sim p(\theta)$

$X_i \sim \dfrac{\theta^x e^{-\theta}}{x!} = f(x_i;\theta)$; $x = 0, 1, 2, 3, \ldots$ (p 94)

MGF = $e^{\theta(e^t-1)}$

MGF OF $\sum_{k=1}^{n} X_k = e^{n\theta(e^t-1)}$

$$\Rightarrow \sum_{k=1}^{n} X_k \sim \frac{(n\theta)^{(\Sigma x)} e^{-n\theta}}{(\Sigma x)!} = g[\Sigma X_k ; \theta]$$

$$\prod_{k=1}^{n} f(x_i;\theta)$$

$$= \frac{\theta^{\Sigma x} e^{-n\theta}}{\prod_{k=1}^{n} x_k!} = \frac{(\Sigma x)!}{\prod_{k=1}^{n} x_k!} g[\Sigma x_k ; \theta]$$

THUS, BY THEM 1, $Y = \sum^{n} X_k$ IS A SUFFICIENT STATISTIC FOR θ.

(7-8) $f(x;\theta) = \frac{1}{\theta}$; $0 < x < \theta$; $\theta > 0$

$(x_1, \dots x_n)$

$Y_n = \max(X_i)$

$$g(Y_n;\theta) = n\, F(Y_n;\theta)^{n-1}\, f(Y_n;\theta)$$

$F(x;\theta) = x/\theta$

$$= \frac{n}{\theta^n} Y_n^{n-1}$$

$$\prod_{i=1}^{n} f(x_i;\theta) = \frac{1}{\theta^n}$$

$$= \underbrace{\left(\frac{n}{\theta^n} Y_n^{n-1}\right)}_{K_1} \underbrace{\left(\frac{1}{n Y_n^{n-1}}\right)}_{K_2}$$

$f(x;\theta) = Q(\theta)\, M(x)$ $\quad 0 < x < \theta$; $\theta > 0$

$$F(x;\theta) = Q(\theta) \int_0^x M(x)\,dx$$

$$g(Y_n;\theta) = n\, Q^n(\theta)\, M(y) \int_0^x M(y)\,dy$$

$$\prod_{i=1}^{n} f(x_i,\theta) = Q^n(\theta) \prod_{i=1}^{n} M(x_i)$$

$$= n\, Q^n(\theta) M(y) \int_0^y M(x)\,dx$$

$$\times \quad \frac{\prod^n M(x_i)}{n\, M(y) \int_0^y M(t)\,dt}$$

(7-9) $X_k \sim f(x_k;\theta) = (1-\theta)^x\theta$; $x = 0, 1, 2, \ldots$

$0<\theta<1$

$$Y = \sum^n X_k$$

$$M_X(t) = \frac{\theta}{[1-(1-\theta)e^t]} \quad \Leftarrow p. 90$$

$$M_Y(t) = \frac{\theta^n}{[1-(1-\theta)e^t]^n} \Leftarrow \text{NEGATIVE BINOMIAL (p. 90)}$$

$$\Rightarrow g[Y;\theta] = \binom{Y+n-1}{n-1}\theta^n(1-\theta)^Y \; ; Y = 0, 1, \ldots$$

$$\prod_{K=1}^{n} f(x_K;\theta) = (1-\theta)^{\Sigma x}\theta^n = (1-\theta)^Y\theta^n$$

$$= g[Y;\theta] \frac{1}{\binom{Y+n-1}{n-1}}$$

THUS BY THEM 1, $Y = \sum X_K$
IS SUFFICIENT STATISTIC FOR θ.

(7-10) $X_K \sim f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$; $0 < x, \theta < \infty$

$Y = \sum^{n} X_K$

$f(x;\theta)$ IS A GAMMA DISTRIBUTION WITH

$\beta = \theta$ AND $\alpha = 1$ (p. 99)

MGF OF $X_K = \frac{1}{(1-\beta t)^{\alpha}} = \frac{1}{(1-\theta t)}$

$\Rightarrow$ MGF OF $\sum X_K = \frac{1}{(1-\theta t)^n}$

$\Rightarrow \sum X_K$ IS GAMMA WITH $\beta = \theta$, $\alpha = n$

$$\Rightarrow \sum X_K \sim g[Y;\theta] = \frac{1}{\Gamma(n)\theta^n} Y^{n-1} \underline{e^{-Y/\theta}}$$

NOW

$$\prod_{K=1}^{n} f(x_K;\theta) = \frac{1}{\theta^n} e^{-\frac{\sum X_K}{\theta}} = \frac{1}{\theta^n} e^{-Y/\theta}$$

$$= g[Y;\theta] \qquad \frac{\Gamma(n)}{Y^{n-1}}$$

AGAIN, BY THEM 1, $\sum_{K}^{n} X_K$ IS A SUFFICIENT STATISTIC FOR θ.

(7-11) $X_K \sim \beta[\alpha = \theta > 0, \beta = 2]$

$$= \frac{\Gamma(\theta+2)}{\Gamma(\theta)\Gamma(2)} X_K^{\theta-1}(1-X_K) \quad \Leftarrow p.\ 134\ \#4.25$$

$$= \frac{\Gamma(\theta+2)}{\Gamma(\theta)} X_K^{\theta-1}(1-X_K) \quad ; 0 < X_K < 1$$

$$\prod_{K=1}^{n} f(X_K;\theta)$$

$$= \left[\frac{\Gamma(\theta+2)}{\Gamma(\theta)}\right]^n \prod_{K=1}^{n}(X_K^{\theta-1}) \prod_{K=1}^{n}(1-X_K)$$

$$= \left[\frac{\Gamma(\theta+2)}{\Gamma(\theta)}\right]^n (X_1 \cdot X_2 \cdots X_n)^{\theta-1} \prod_{K=1}^{n}(1-X_K)$$

SINCE $0 < X_K < 1 \Rightarrow \prod_{K=1}^{n}(1-X_K) > 0$

LET $k_2 = \prod_{K=1}^{n}(1-X_K)$

ALSO

$k_1 = \left[\frac{\Gamma(\theta+2)}{\Gamma(\theta)}\right]^n (X_1 \cdot X_2 \cdots X_n)^{\theta-1} > 0$

THUS, BY THEM. 2, $X_1 X_2 \cdots X_n$ IS A SUFFICIENT STATISTIC FOR θ.

CAUCHY

(7-12) $X_K \sim \dfrac{1}{\pi[1+(x-\theta)^2]}$; $-\infty < x, \theta < \infty$

$$\prod_{K=1}^{n} \frac{1}{\pi[1-(x_K-\theta)^2]} = \pi^{-n} \frac{1}{\prod_{K=1}^{n}[1-(x_K-\theta)^2]}$$

$$= \frac{\pi^{-n}}{\prod_{K=1}^{n}[1-(x_K-\theta)^2]} \leftarrow \text{JOINT PDF}$$

THERE IS NO APPARENT WAY THIS JOINT PDF CAN BE PUT IN THE FORM OF THEM. 2. NOW, θ IS THE MEDIAN OF THE pdf OF X_K WHICH IS SYMMETRIC ABOUT θ. BUT THE MEAN IS UNDEFINED AND THE VARIANCE IS INFINATE. IT SEEMS TO FOLLOW FROM THIS WIERD BEHAVIOR THAT THERE EXISTS NO SUFFICIENT STATISTIC FOR THE MEDIAN θ. (THIS FOLLOWS FROM THE "ONLY IF" PART OF "IF AND ONLY IF" IN THEM. 2)

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(7-14) $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta} \qquad 0 < x, \theta < \infty$

$$Y_1 = X_1 + X_2 \qquad Y_2 = X_2$$

$$X_1 = Y_1 - Y_2 \qquad X_2 = Y_2$$

$$|J| = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$\Rightarrow f(x_1, x_2; \theta) = \frac{1}{\theta^2} e^{-(x_1+x_2)/\theta} \; ; \; 0 < x_1, x_2 < \infty$$

$$\Rightarrow f(y_1, y_2; \theta) = \frac{1}{\theta^2} e^{-[(y_1 - y_2) + y_2]/\theta}$$

$$= \frac{1}{\theta^2} e^{-y_1/\theta} \; ; \quad 0 < y_2 < y_1 < \infty$$

(a) $f_1(y_2) = \int_{y_2}^{\infty} \frac{1}{\theta^2} e^{-y_1/\theta} \, dy_1$

$$= \frac{1}{\theta^2} \, -\theta e^{-\frac{y_1}{\theta}} \Big|_{y_2}^{\infty}$$

$$= -\frac{1}{\theta} \left[0 - e^{-y_2/\theta} \right]$$

$$= \frac{1}{\theta} e^{-y_2/\theta} \; ; \quad 0 < y_2 < \infty$$

$$E[Y_2] = \frac{1}{\theta} \int_0^{\infty} y_2 \left[e^{-y_2/\theta} \right] dy_2$$

$$\cancel{u = y_2} \qquad \cancel{dv = \left[1 - e^{-y_2/\theta}\right] dy_2}$$

$$\cancel{du = dy_2} \qquad \cancel{v = \left(y_2 + \theta e^{-y_2/\theta}\right)}$$

THE MEAN OF THE ONE SIDED EXPONENTIAL PDF IS θ

$$\Rightarrow E[Y_2] = \theta$$

THE VARIANCE IS θ^2

$$\Rightarrow Var.[Y_2] = \theta^2$$

(b) $h(y_2 | y_1) = \dfrac{f(y_1, y_2)}{f(y_1)}$

$$f(y_1) = \int_0^{y_1} \frac{1}{\theta^2} e^{-y_1/\theta} \, dy_2$$

$$= \frac{y_1}{\theta^2} e^{-y_1/\theta} \qquad ; \; 0 < y_1 < \infty$$

$\Longrightarrow$

$$h(Y_2 | Y_1) = \frac{\frac{1}{\theta^2} e^{-Y_1/\theta}}{\frac{Y_1}{\theta^2} e^{-Y_1/\theta}} \qquad 0 < Y_2 < Y_1 < \infty$$

$$= \frac{1}{Y_1} \qquad 0 < Y_2 < Y_1 < \infty$$

$$E[Y_2 / Y_1] = \int_{Y_2} Y_2 h(Y_2 | Y_1)\, dY_2$$

$$= \int_0^{Y_1} \frac{Y_2}{Y_1}\, dY_2$$

$$= \frac{1}{Y_1} \frac{1}{2} Y_1^2 = \frac{1}{2} Y_1 = \phi(Y_1)$$

$$E[\phi(Y_1)] = \int_{Y_1} \phi(Y_1) f_1(Y_1)\, dY_1$$

$$= \int_0^{\infty} \left(\frac{1}{2} Y_1\right)\left(\frac{Y_1}{\theta^2}\right) e^{-Y_1/\theta}\, dY_1$$

$$= \frac{1}{2\theta^2} \int_0^{\infty} Y_1^2 e^{-Y_1/\theta}\, dY_1$$

$$Y_1' = \frac{Y_1}{\theta} \Rightarrow Y_1 = \theta Y_1'$$

$$= \frac{1}{2\theta^2} \int_0^{\infty} (\theta Y_1')^2 e^{-Y_1'}\, \theta\, dY_1'$$

$$= \frac{\theta}{2} \int_0^{\infty} Y_1'^{\,3-1} e^{-Y_1'}\, dY_1'$$

$$= \frac{\theta}{2} \Gamma(3)$$

$$= \frac{\theta}{2}(2!) = \theta$$

$$\mathrm{Var}\, \phi(Y_1) = E[\{\phi(Y_1) - \theta\}^2]$$

$$= E[\phi^2] - \theta^2$$

$$E[\phi^2] = \int_0^{\infty} \left(\frac{1}{4} Y_1^2\right) \frac{Y_1}{\theta^2} e^{-Y_1/\theta}\, dY_1$$

$$= \frac{1}{4\theta^2} \int_0^{\infty} Y_1^3 e^{-Y_1/\theta}\, dY_1$$

$$Y_1' = Y_1/\theta$$

$$\Rightarrow E[\phi^2] = \frac{1}{4\theta^2} \int_0^{\infty} \theta^3 Y_1'^3 e^{-Y_1'} (\theta\, dY_1')$$

$$= \frac{\theta^2}{4} \int_0^{\infty} Y_1^{4-1} e^{-Y_1}\, dY_1$$

$$= \frac{\theta^2}{4} \Gamma(4) = \frac{\theta^2}{4} 3! = \frac{6}{4}\theta^2 = \frac{3}{2}\theta^2$$

$$\Rightarrow \mathrm{Var}\, \phi(Y_1) = \frac{3}{2}\theta^2 - \theta^2 = \frac{1}{2}\theta^2$$

AS EXPECTED, $\mathrm{Var}\, \phi(Y_1) < \mathrm{Var}\, Y_2$

(7-15) $f(x,y) = \frac{2}{\theta^2} e^{-(x+y)/\theta}$; $0<x<y<\infty$

(a) $f_1(y) = \frac{2}{\theta^2} \int_0^y e^{-(x+y)/\theta} dx$

$= \frac{2}{\theta^2} e^{-y/\theta} \int_0^y e^{-x/\theta} dx$

$= \frac{2}{\theta^2} e^{-y/\theta} \; +\theta e^{-x/\theta} \Big|_y^0$

$= \frac{2}{\theta} e^{-y/\theta} [1 - e^{-y/\theta}]$

$= \frac{2}{\theta} e^{-y/\theta} - \frac{2}{\theta} e^{-2y/\theta}$; $0<y<\infty$

$E[Y] = 2\int_0^\infty \frac{y}{\theta} e^{-y/\theta} dy - \int_0^\infty \left(\frac{2}{\theta}\right) y e^{-y/(\theta/2)} dy$

$= 2\theta - \frac{\theta}{2} = \frac{3}{2}\theta$

$E[Y^2] = \frac{2}{\theta}\int_0^\infty y^2 e^{-y/\theta} dy - \frac{2}{\theta}\int_0^\infty y^2 e^{-\frac{2y}{\theta}} dy$

$y_1 = +\frac{y}{\theta}$ $\quad y_2 = \frac{+2y}{\theta}$

$E(Y^2) = \frac{2}{\theta}\theta^3 \int_0^\infty y_1^{3-1} e^{-y_1} dy_1 - \frac{2}{\theta}\left(\frac{\theta}{2}\right)^3 \int_0^\infty y_2^{3-1} e^{-y_2} dy_2$

$= 2\theta^2 \Gamma(3) - \frac{2\theta^2}{8}\Gamma(3)$

$= 4\theta^2 - \frac{1}{2}\theta^2 = \frac{7}{2}\theta^2$

$Var\, Y = \frac{7}{2}\theta^2 - \frac{9}{4}\theta^2 = \frac{5}{4}\theta^2$

(b) $h(y|x) = \frac{f(x,y)}{f(x)}$

$f(x) = \frac{2}{\theta^2} \int_x^\infty e^{-(x+y)/\theta} dy$

$= \frac{2}{\theta^2} e^{-x/\theta} \int_x^\infty e^{-y/\theta} dy$

$= \frac{2}{\theta} e^{-x/\theta} e^{-y/\theta} \Big|_\infty^x$

$= \frac{2}{\theta} e^{-x/\theta} [e^{-x/\theta}]$

$= \frac{2}{\theta} e^{-2x/\theta}$; $0<x<\infty$

$\Rightarrow h(y|x) = \frac{\frac{2}{\theta^2} e^{-x/\theta} e^{-y/\theta}}{\frac{2}{\theta} e^{-x/\theta} e^{-x/\theta}}$; $0<x<y<\infty$

$= \frac{1}{\theta} e^{-y/\theta} e^{x/\theta}$ $\Longrightarrow$

$$h(y|x) = \frac{1}{\theta} e^{-y/\theta} e^{x/\theta} \quad 0 < x < y < \infty$$

$$E[Y|x] = \frac{1}{\theta} e^{x/\theta} \int_x^\infty y e^{-y/\theta} dy$$

$$u = y \qquad dv = e^{-y/\theta}$$

$$du = dy \qquad v = -\theta e^{-y/\theta}$$

$$E[Y|x] = \frac{1}{\theta} e^{x/\theta} \left[-y\theta e^{-y/\theta} \Big|_x^\infty + \theta \int_x^\infty e^{-y/\theta} dy \right.$$

$$= \frac{1}{\theta} e^{x/\theta} \left[-(0 - x\theta e^{-x/\theta}) - \theta^2 e^{-y/\theta} \Big|_x^\infty \right.$$

$$= \frac{1}{\theta} e^{x/\theta} \left[x\theta e^{-x/\theta} + \theta^2 e^{-x/2} \right]$$

$$= x + \theta$$

$$E(x+\theta) = \frac{2}{\theta} \int_0^\infty (x+\theta) e^{-2x/\theta} dx$$

$$= \frac{2}{\theta} \int_0^\infty x e^{-x(\theta/2)} dx + 2 \int_0^\infty e^{-2x/\theta} d\theta$$

$$= \frac{\theta}{2} + 2 \left(\frac{\theta}{2}\right) \int_0^\infty \frac{2}{\theta} e^{-2x/\theta} d\theta$$

$$= \frac{\theta}{2} + \theta (1) = 3\theta/2$$

$$Var[x+\theta] = E[(x+\theta)^2] - E[(x+\theta)]^2$$

$$E((x+\theta)^2) = \frac{2}{\theta} \int_0^\infty (x+\theta)^2 e^{-2x/\theta} dx$$

$$u = (x+\theta)^2 \qquad dv = \frac{2}{\theta} e^{-\frac{2x}{\theta}} dx$$

$$du = 2(x+\theta) dx \qquad v = -e^{-\frac{2x}{\theta}}$$

$$\Rightarrow E[(x+\theta)^2] = +(x+\theta)^2 e^{-\frac{2x}{\theta}} \Big|_\infty^0 + 2 \int_0^\infty (x+\theta) e^{-\frac{2x}{\theta}} dx$$

$$= \theta^2 + 2 \int_0^\infty (x+\theta) e^{-2x/\theta} dx$$

$$u = 2(x+\theta) \qquad dv = e^{-2x/\theta} dx$$

$$du = 2dx \qquad v = -\frac{\theta}{2} e^{-2x/\theta}$$

$$\Rightarrow E[(x+\theta)^2] = \theta^2 + \cancel{2}(x+\theta)\frac{\theta}{\cancel{2}} e^{-2x/\theta} \Big|_\infty^0 + \int_0^\infty \theta e^{-2x/\theta} d\theta$$

$$= \theta^2 + \theta^2 + \frac{\theta^2}{2} \int_0^\infty \frac{2}{\theta} e^{-2x/\theta} dx$$

$$= \frac{5}{2}\theta^2$$

$$Var(x+\theta) = \frac{5}{2}\theta^2 - \frac{9\theta^2}{4}$$

$$= \frac{1}{4}\theta^2$$

(7.16) $\gamma(Y_1) = E[\phi(Y_1) | Y_2] = E[E(Y_2/Y_1) | Y_2]$

$= E\left[\left\{\int_{Y_2} Y_2\, h(Y_2/Y_1)\, dY_2\right\} \Big| Y_2\right]$

SINCE Y_1 IS A SUFFICIENT STATISTIC, IT FOLLOWS THAT $h(Y_2/Y_1)$ DOES NOT DEPEND ON θ. THUS, $\phi(Y_1)$ IS A FUNCTION OF Y_1 ALONE, AND

$\gamma(Y_1) = E[\phi(Y_1) | Y_2] = E[\phi(Y_1)] = \theta$

ALTHOUGH IT IS TRUE THAT

$E[\gamma(Y_1)] = E[\theta] = \theta E[1] = \theta$,

$\gamma(Y_1)$ IS DETERMINISTIC. THAT IS IT IS NOT A STATISTIC. ALTERNATIVELY, WE COULD WRITE

$E[E(Y_2 | X_1 = 1) | X_2 = Y_2]$

$= E[E[X_2 = Y_2 | X_1 = Y_1]]$

$= E[X_2] = \theta$

$\gamma(Y_2) = E[\phi(Y_1) | Y_2]$ IS A FUNCTION OF Y_2 WHICH DEPENDS ON θ.

$\therefore$ $\gamma(Y_2)$ DEPENDS ON AN UNKNOWN PARAMETER AND IS THUS NOT A STATISTICS

(7-17)

$$p(x,\theta) = \binom{2}{k}\theta^k(1-\theta)^{n-k} \quad \theta = 0, 1, 2$$

$$E[U(X)] = 0$$

$$= U(0)(1-\theta)^2 + 2U(1)\theta(1-\theta) + U(2)\theta^2$$

$$= U(0)(1-2\theta+\theta^2) + 2U(1)(\theta-\theta^2) + U(2)\theta^2$$

$$= \theta^2[U(0) - 2U(1) + U(2)] + \theta[-2U(0) + 2U(1)] + U(0)$$

WE REQUIRE $U(0) = 0$

$$U(0) - 2U(1) + U(2) = -2U(1) + U(2) = 0$$

$$\Rightarrow U(2) = 2U(1)$$

$$+2U(0) = 0 = 2U(1) \Rightarrow U(1) = 0$$

$$\Rightarrow U(2) = 0$$

QED

(7-18) $$f(x;\theta) = \frac{\theta^x e^{-\theta}}{x!} \quad x = 0, \ldots \infty, \; \theta > 0$$

$$E[U(x)] = 0 = \sum_{x=0}^{\infty} \frac{U(x)\,\theta^x e^{-\theta}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{U(x)\,\theta^x}{x!}$$

SINCE A TAYLOR SERIES EXPANSION IS UNIQUE $\Rightarrow U(x)\,\theta^x = 0 \;\forall x$

$$\Rightarrow U(x) = 0 \quad \forall x \;\; \forall \theta > 0$$

QED

(7-19) $g(Y_n;\theta) = n\,[F(Y_n)]^{n-1} f(Y_n)$

$$F(Y_n) = \begin{cases} \frac{Y_n}{\theta} & ; \; 0 < Y_n < \theta \\ 1 & ; \; Y_n > \theta \\ 0 & ; \; Y_n < 0 \end{cases}$$

$$\Rightarrow g(Y_n;\theta) = \begin{cases} 0 & ; \; Y_n < 0 \\ n\left(\frac{Y_n}{\theta}\right)^{n-1}\frac{1}{\theta} = \frac{n\,Y_n^{n-1}}{\theta^n} & ; \; 0 < Y_n < \theta \\ 0 & ; \; Y_n > \theta \end{cases}$$

$$E[U(Y_n)] = \int_0^\theta U(Y_n)\, n \,\frac{Y_n^{n-1}}{\theta^n}\, dY_n$$

$$= \frac{n}{\theta^n}\int_0^\theta U(Y_n)\, Y_n^{n-1}\, dY_n = 0$$

OR

$$\int_0^\theta U(Y_n)\, Y_n^{n-1}\, dY_n = 0$$

FOR $U \neq 0$, THIS RELATION REQUIRES U TO BE A FUNCTION OF θ, WHICH IS NOT ALLOWABLE. THAT IS, $\exists$ NO (CONTINUOUS) FUNCTION $U(Y_n)$ THAT IS ORTHOGONAL TO Y_n^{n-1} ON THE INTERVAL $(0,\theta)$ WHERE θ IS ARBITRARY.

QED

(7-20) $f(x;\theta) = Q(\theta)M(x)$; $0<x<\theta$, $\theta>0$

$$g(y_n;\theta) = n\,F^{n-1}(y_n)\,f(y_n) \quad \left(\text{FROM FINAL}\right)$$

$$= n\,Q^n(\theta)\,M(x)\left[\int_0^{y_n} M(x)\,dx\right]^{n-1} ;\ 0<y_n<\theta$$

$$E[u(Y_n)] = n\,Q^n(\theta)\int_0^{\theta} u(x)M(x)\left[\int_0^{y_n}(M(x)\,dx)\right]^{n-1}dx$$

$$= 0$$

$$= \int_0^{\theta} u(x)\,M(x)\left[\int_0^{x} M(x)\,dx\right]^{n-1}dx$$

$$\frac{d}{d\theta}E[u(Y_n)] = 0 = u(\theta)\,M(\theta)\left[\int_0^{\theta} M(x)\,dx\right]^{n-1}$$

BUT

$$\int_0^{\theta} Q(\theta)M(x)\,dx = 1 \Rightarrow \int_0^{\theta} M(x)\,dx = \frac{1}{Q(\theta)}$$

$$\Rightarrow 0 = u(\theta)\;M(\theta)\left[\frac{1}{Q(\theta)}\right]^{n-1}$$

THUS, WE REQUIRE $u(\theta) = 0$

BUT, u IS NOT ALLOWED TO

BE A FUNCTION OF $\theta \Rightarrow u(x) = 0$

(7-22)

(a) $f(x;\theta) = \frac{1}{2\theta}$; $-\theta < x < \theta$

$E[u(x)] = 0 = \frac{1}{2\theta}\int_{-\theta}^{\theta} u(x)\,dx$

LET $u(x)$ BE ANY ODD FUNCTION.*

FOR EXAMPLE, $u(x) = x$

(b)

$n(0,\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{x^2}{2\theta}}$

$E[u(x)] = \frac{1}{\sqrt{2\pi\theta}} \int_{-\infty}^{\infty} u(x)\, e^{-\frac{x^2}{2\theta}} dx$

AGAIN, LET $u(x)$ BE ANY ODD FUNCTION.* FOR EXAMPLE $u(x) = x$

* FOR WHICH THE INTEGRAL CONVERGES

$(7\text{-}22) \quad b(n,\theta) = \binom{n}{k}\theta^k(1-\theta)^{n-k}$

$$E[U(x)] = \sum_{k=0}^{n}\binom{n}{k}U(k)\theta^k(1-\theta)^{n-k} = 0$$

WE HERE HAVE AN n^{TH} ORDER POLYNOMIAL, THE COEFFICIENTS OF EACH POWER OF θ MUST BE IDENTICALLY ZERO. U(K) IS SIMPLY A LINEAR SUM OF THE POLYNOMIAL COEFFICIENTS AND THUS, MUST ALSO BE ZERO $\forall$ K.

FIND $\hat{f}(\mu)$, σ^2 KNOWN

$$\hat{f}(\mu) = f(\bar{x}) + \sum_{k=1}^{\infty} \frac{(-1)^k f^{(2k)}(\bar{x})}{k!} \left(\frac{\sigma^2}{2n}\right)^k$$

① $f(\mu) = e^{-\mu} \Rightarrow f^{(2k)}(\mu) = e^{-\mu}$

$$\hat{f}(\mu) = e^{-\bar{x}} + e^{-\bar{x}} \sum_{k=1}^{\infty} \frac{(-\sigma^2/2n)^k}{k!}$$

$$= e^{-\bar{x}} \sum_{k=0}^{\infty} \frac{(-\sigma^2/2n)^k}{k!}$$

$$= e^{-\bar{x}} e^{-\sigma^2/2n}$$

② $f(\mu) = \mu^3 - 4\mu^2 + \mu$

$f'(\mu) = 3\mu^2 - 8\mu + 1$

$f''(\mu) = 6\mu - 8$

$f'''(\mu) = 6$

$f''''(\mu) = 0$

$$\Rightarrow \hat{f}(\mu) = (\bar{x}^3 - 4\bar{x}^2 + \bar{x}) - \frac{\sigma^2}{2n}(6\bar{x} - 8)$$

BOB MARKS
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(7-23) (a) $X_i \sim f(x;\theta) \qquad Y = \sum^{n} X_i$

$f(x;\theta) = \theta^x (1-\theta^x) \qquad ; x = 0,1 \quad 0<\theta<1$

(A BERNOULLI TRIAL)

$\Rightarrow f(Y;\theta) = b(n,\theta)$

WE SHOWED IN (7-22) THAT $b(n,\theta)$ IS COMPLETE AND IN (7-10) THAT Y IS SUFFICIENT STATISTIC FOR θ.

NOW $E[Y] = n\theta$

$\Rightarrow E[\frac{Y}{n} = \bar{X}] = \theta$

$\frac{Y}{n}$ IS THE BEST STATISTIC FOR θ

(b) $f(x;\theta) = \theta^x e^{-\theta}/x!$

$M_x(t) = e^{\theta(1-e^t)} \Rightarrow M_Y(t) = e^{n\theta(1-e^t)}$

$\Rightarrow f(Y;\theta) = (n\theta)^Y e^{-n\theta}/Y!$

WE SHOWED IN (7-18) THAT $f(Y;n\theta)$ IS COMPLETE AND IN (7-7) THAN Y IS A SUFFICIENT STATISTIC FOR θ

$E[Y] = n\theta \Rightarrow E[\frac{Y}{n} = \bar{X}] = \theta$

$\frac{Y}{n}$ IS THE BEST STATISTIC FOR θ.

(7-24) $Y_1 = \min\{X_1 \ldots X_n\}$

$P_r[Y > Y] = [P_r[X_i > Y]]^n$

$1 - F(Y) = [1 - F_x(Y)]^n$

$\Rightarrow F(Y) = 1 - [1 - F_x(Y)]^n$

$f(Y) = \frac{d}{dY} F(Y) = n f_x(Y)[1 - F_x(Y)]^{n-1}$

$f_x(Y) = e^{-(x-\theta)}$; $\theta < x < \infty$

$F_x(Y) = 1 - e^{-(x-\theta)}$; $\theta < x < \infty$

$\Rightarrow f(Y) = n e^{-(Y-\theta)} [e^{-(Y-\theta)}]^{n-1}$

$= n e^{-n(Y-\theta)}$; $\theta < Y < \infty$

IS IT SUFFICIENT?

$\prod_{i=1}^{n} f_x(x_i; \theta) = \prod_{i=1}^{n} e^{-(x_i - \theta)}$

$= e^{n\theta - \sum_n x_i}$

$= \underbrace{e^{n\theta} e^{-Y}}_{K_1} \underbrace{e^{-\sum_{n \neq Y} x_i}}_{K_2}$

NOTE: ALL $X_i \neq Y$ HAVE DOMAIN $X_i \geq Y$ AND ARE THUS θ INDEP.

BY THEM 2 ON p. 219, Y IS SUFFICIENT

NOW $E[U(Y)] = \int_\theta^\infty U(Y)\, n e^{-n(Y-\theta)} dY$

$= \int_0^\infty U(Y+\theta)(n e^{-nY}) dY = 0$

FOR $U(Y) \neq 0$, U MUST DEPEND ON θ.

THUS, BY DEF 4, p. 227, $f_Y(Y;\theta)$ IS COMPLETE

$E[Y] = n \int_\theta^\infty Y e^{-n(Y-\theta)} dY$

$= n \int_0^\infty (Y+\theta) e^{-nY} dY$

$= n \int_0^\infty Y e^{-nY} dY + n\theta \int_0^\infty e^{-nY} dY$

$= [n \int_0^\infty Y e^{-Y/(\frac{1}{n})} dY] + \theta [n \int_0^\infty e^{-Y/(\frac{1}{n})} dY]$

$= \frac{1}{n} + \theta$

$\Rightarrow E[Y - \frac{1}{n}] = \theta$

THUS $(Y - \frac{1}{n})$ IS BEST STATISTIC FOR θ

(7-25) $f(x_i;\theta) = \frac{1}{\theta}$; $x = 1, \ldots \theta$

$Y = \max[x_1 \ldots x_\theta)$

$F_Y^{(Y)} = F_x^n(Y)$

$F_x(Y) = \frac{Y}{\theta}$

$\Rightarrow F_Y(Y) = \left(\frac{Y}{\theta}\right)^n$; $Y = 1, \ldots, \theta$

$f(Y) = F_Y(Y) - F_Y(Y-1)$

$= \frac{1}{\theta^n}\left[Y^n - (Y-1)^n\right]$

IS Y SUFF ?

$\prod_{i=1}^{n} f(x_i;\theta] = \prod_{i=1}^{n}\frac{1}{\theta}$; $x_i = 1, 2, \ldots \theta$

$= \frac{1}{\theta^n}$; $x_i = 1, 2, \ldots, \theta$

$= \frac{1}{\theta^n}$; $\begin{cases} Y = \max\{X_i\} = 1, 2, \ldots, \theta \\ X_i = 1, 2, \ldots, Y \end{cases}$

$\Rightarrow$ SUFFICIENT

COMPLETE ?

$E[u(Y)] = \frac{1}{\theta^n}\sum_{Y=1}^{\theta} u(Y)\left[Y^n - (Y-1)^n\right] = 0$

THIS (THE Σ) IS A $(n-1)$ ORDER POLYNOMIAL IN θ WHICH, IN GENERAL, CAN BE ZERO FOR ZERO COEFFICIENTS $\Rightarrow u(Y) = 0$

$$E\left[\frac{Y^{n+1} - (Y-1)^{n+1}}{Y^n - (Y-1)^n}\right] = \sum_{Y=1}^{\theta}\left[\frac{Y^{n+1} - (Y-1)^{n+1}}{Y^n - (Y-1)^n}\right]\frac{1}{\theta^n}\left[Y^n - (Y-1)^n\right]$$

$$= \frac{1}{\theta^n}\sum_{Y=1}^{\theta}\left[Y^{n+1} - (Y-1)^{n+1}\right]$$

$$= \frac{1}{\theta^n}\left[\theta^{n+1}\right] = \theta$$

$\left(\text{SINCE } \underbrace{\frac{1}{\theta^n}\sum_{Y=1}^{\theta}\left[Y^n - (Y-1)^n\right]}_{\text{A pdf}} \equiv 1\right)$

(7-26) $\omega \in \Omega$

$f(x;\theta)$; $\theta \in \Omega$

(a) SHOW Y IS SUFF $\forall \theta \in \Omega \Rightarrow Y$ IS SUFF $\forall \theta \in \omega$

THIS SEEMS TO FOLLOW TRIVIALLY SINCE,

Y IS SUFF. $\underline{\underline{\forall}}$ Ω AND $\omega \in \Omega$

(b) SHOW g IS COMPL $\forall \theta \in \omega$

$\Rightarrow g$ IS COMPL $\forall \theta \in \Omega$

THIS IS <u>NOT</u> ALWAYS TRUE

A COUTEREXAMPLE:

ON PG 226, IT WAS SHOWN

$$f(x;\theta) = \frac{1}{\theta} \quad ; 0 < x < \theta$$
$$; \ 0 < \theta < \infty$$

IS COMPLETE. HERE $\Omega = \{\theta \mid 0 < \theta < \infty\}$

CONSIDER $\omega \in \Omega$ WHERE

$$\omega = n \ \ni n = 1, 2, 3, \ldots$$

LET $u(x) = \sin 2\pi x$

THEN

$$E[u(x)] = \frac{1}{\theta}\int_0^\theta \sin 2\pi x \, dx$$

$$= \frac{1}{n}\int_0^n \sin 2\pi x \, dx$$

$$= \frac{-1}{n} \frac{1}{2\pi} \cos 2\pi x \Big|_0^n$$

$$= \frac{-1}{2\pi n}\left[\cos 2\pi n - \cos 0\right]$$

BUT $\cos 2\pi n = \cos 0 = 1$

$$\Rightarrow E[u(x)] = 0$$

QED

(BUT THIS WASN'T WHAT WAS ASKED FOR)

(7-27) $f(x;\theta) = Q(\theta)\, M(x) \;;\; 0 < x < \theta$

LET Y_n BE THE LARGEST STATISTIC FROM n SAMPLES. WE SHOWED IN PROB 7.20 p 228 THAT Y_n IS COMPLETE & SUFFICIENT. NOW

$$F_Y(Y) = F_X^n(Y)$$

$$\Rightarrow f_Y(Y) = n\, f_X(Y)\, F_X^{n-1}(Y)$$

$$F_X(Y) = Q(\theta) \int_0^Y M(x)\,dx \;;\; 0 < Y < \theta$$

$$\Rightarrow f_Y(Y) = n\, Q^n(\theta)\, M(Y) \left[\int_0^Y M(x)\,dx\right]^{n-1}$$

NOW, WE WISH TO FIND $\phi(Y) \ni$

$$\theta = E[\phi(Y)] = n\, Q^n(\theta) \int_0^\theta \phi(Y)\, M(Y) \left[\int_0^Y M(x)\,dx\right]^{n-1} dY$$

OR

$$n \int_0^\theta \phi(Y)\, M(Y) \left[\int_0^Y M(x)\,dx\right]^{n-1} dY = \frac{\theta}{Q^n(\theta)}$$

DIFFERENTIATE W.R.T. θ:

$$n\,\phi(\theta)\, M(\theta) \left[\int_0^\theta M(x)\,dx\right]^{n-1} = \frac{d}{d\theta} \frac{\theta}{Q^n(\theta)}$$

$$n\,\phi(\theta)\, M(\theta) \left[1/Q^{n-1}(\theta)\right] = \frac{d}{d\theta} \frac{\theta}{Q^n(\theta)}$$

$$\Rightarrow \phi(\theta) = \frac{1}{M(\theta)\, n}\, Q^{n-1}(\theta)\, \frac{d}{d\theta} \frac{\theta}{Q^n(\theta)}$$

$$= \frac{1}{n M(\theta)}\, Q^{n-1}(\theta) \left[\frac{1}{Q^n(\theta)} - \frac{n\theta\, Q'(\theta)}{Q^{n+1}(\theta)}\right]$$

$$= \frac{1}{n M(\theta)}\, \frac{Q^{n-1}(\theta)\left[Q(\theta) - n\theta\, Q'(\theta)\right]}{Q^{n+1}(\theta)}$$

$$= \frac{1}{n M(\theta)}\, \frac{Q(\theta) - n\theta\, Q'(\theta)}{Q^2(\theta)}$$

OR

$$\phi(Y) = \frac{Q(Y) - nY \dfrac{dQ(Y)}{dY}}{n\, M(Y)\, Q^2(Y)}$$

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(7-28) $f(x;\theta) = \dfrac{1}{6\theta^4} x^3 e^{-x/\theta}$; $0 < x, \theta < \infty$

$$= e^{\ln \frac{1}{6\theta^4}} e^{\ln x^3} e^{-x/\theta}$$

$$= e^{\underset{p(\theta)}{\left(-\frac{1}{\theta}\right)} \underset{K(x)}{x} + \underset{S(x)}{3 \ln x} - \underset{q(\theta)}{\ln 6\theta^4}}$$

$Y = \Sigma_{i=1}^{k} X_i$ IS COM. SUFF. STAT.

$f(x;\theta)$ IS A GAMMA DISTRIBUTION

WITH A $\alpha = 4$, $\beta = \theta$

$$M_x(t) = \frac{1}{(1-\beta t)^\alpha} = \frac{1}{(1-\theta t)^4}$$

$$\Rightarrow M_Y(t) = \frac{1}{(1-\theta t)^{4n}} = [M_x(t)]^n$$

$$f(y;\theta) = \frac{1}{\Gamma(4n)\,\theta^{4n}} x^{4n-1} e^{-x/\theta}$$

$$E[Y] = \alpha\beta = 4n\theta$$

$$\Rightarrow E\left[\frac{Y}{4n}\right] = \theta$$

$\dfrac{Y}{4n}$ IS BEST STATISTIC

YES, $\dfrac{Y}{4n}$ IS A COMP. SUFF. STAT.

(9-29) $f(x;\theta) = \theta e^{-\theta x}$; $0 < x < \infty$

$K(x) = x$ $\quad \theta > 0$

$f(x;\theta) = e^{-\theta x + \ln\theta}$

$f(x;\theta)$ IS A GAMMA DISTRIBUTION WITH $\alpha = 1$, $\beta = \frac{1}{\theta}$

$\therefore M_X(t) = (1-\beta t)^{-\alpha}$

$= \left(1 - \frac{t}{\theta}\right)^{-1}$

$M_Y(t) = M_X^n(t) = \left(1 - \frac{t}{\theta}\right)^{-n}$

$\Rightarrow Y$ IS GAMMA WITH $\alpha = n$ & $\beta = \frac{1}{\theta}$:

$f(Y;\theta) = \frac{1}{\Gamma(n)\left(\frac{1}{\theta}\right)^n} Y^{n-1} e^{-\theta Y}$

$E[Y] = \alpha\beta = n/\theta$ (THIS WON'T WORK)

ASSUME $\phi(Y) = \frac{n-1}{Y}$

THEN

$E[\phi(Y)] = \frac{\theta^n (n-1)}{(n-1)!} \int_0^\infty Y^{n-2} e^{-\theta Y} dY$

$= \frac{\theta^n}{(n-2)!} \int_0^\infty Y^{n-2} e^{-\theta Y} dY$

LET $\hat{Y} = \theta Y \Rightarrow Y = \frac{\hat{Y}}{\theta}$

$E[\phi(Y)] = \frac{\theta^n}{(n-2)!} \left(\frac{1}{\theta}\right)^{n-1} \int_0^\infty Y^{n-2} e^{-Y} dY$

$\underbrace{\int_0^\infty Y^{(n-1)-1} e^{-Y} dY}_{= \Gamma(n-1) = (n-2)!}$ (p. 99)

$\therefore E[\phi(Y)] = \frac{\theta^n}{(n-2)!} \left(\frac{1}{\theta}\right)^{n-1} (n-2)!$

$= \theta$

(7-30) $f(x;\theta) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}$ ⇐ p. 99

$Y = \sum x_i$

$M_x(t) = \frac{1}{(1-\theta t)^\alpha} \Rightarrow M_Y(t) = \frac{1}{(1-\theta t)^{n\theta}}$

$g(Y;\theta) = \frac{1}{\Gamma(n\alpha)\theta^{n\alpha}} x^{n\alpha-1} e^{-x/\theta}$

NOW $E[Y] = \alpha_Y \beta = n\alpha\beta = n\alpha\theta$

$\bar{X} = Y/n \Rightarrow E[\bar{X}/\alpha] = \theta$

NOW, ASSUME α IS KNOWN:

$E[X_1 | \bar{X}] = E[X_1 / \bar{X}/\alpha]$

$\bar{X}/\alpha$ IS THE BEST STATISTIC FOR θ, THUS, IT IS UNBIASED. THE STATISTIC X_1 IS SUFFICIENT FOR θ (AND X_1/α IS UNBIASED) FROM THEM 4 ON p. 225, LET $E[X_1 | \bar{X}/\alpha] = \phi(\bar{X})$. $\phi(\bar{X})$ IS AN UNBIASED STATISTIC FOR $\theta \ni Var_\phi \le Var_{\bar{X}}$. <u>BUT</u> $\bar{X}/\alpha$ IS THE BEST STATISTIC FOR θ. THUS

$\phi(\bar{X}) = \bar{X}/\alpha$

(7-31)

$$f(x;\theta) = e^{p(\theta)K(x)+S(x)+q(\theta)} \quad a<x<b$$

Now

$$\int_a^b f(x;\theta)dx$$

$$= \int_a^b e^{p(\theta)K(x)+S(x)+q(\theta)} dx = 1$$

$$\frac{d}{d\theta}\int_a^b e^{p(\theta)K(x)+S(x)+q(\theta)} dx$$

$$= \int_a^b \left(p'(\theta)K(x)+q'(\theta)\right)e^{p(\theta)K(x)+S(x)+q(\theta)} dx = 0$$

$$= p'(\theta)\int_a^b K(x)e^{(\quad)}dx + q'(\theta)\int_a^b e^{(\quad)}dx$$

$$= p'(\theta)E[K(x)] + q'(\theta) = 0$$

$$\Rightarrow E[K(x)] = \frac{-q'(\theta)}{p'(\theta)}$$

(7-32) $Y = K(X)$

$$\begin{aligned}
M(t) &= E\left[e^{Yt}\right] \\
&= E\left[e^{K(x)t}\right] \\
&= \int_a^b e^{K(x)t}\, e^{\theta K(x) + S(x) + q(\theta)}\,dx \\
&= \int_a^b e^{(t+\theta)K(x) + S(x) + q(\theta)}\,dx \\
&= e^{q(\theta)} \int_a^b e^{(t+\theta)K(x) + S(x)}\,dx \\
&= e^{q(\theta) - q(\theta+t)} \int_a^b e^{\underbrace{(t+\theta)}_{\theta'}K(x) + S(x) + q\underbrace{(\theta+t)}_{\theta'}}\,dx \\
&= e^{q(\theta) - q(\theta+t)} \quad ; \quad \gamma \le \overbrace{\theta + t}^{\theta'} \le \delta
\end{aligned}$$

(7-33) $E[Y] = E[K(x)] = \theta$

NOW

$$E[e^{K(x)t}] = e^{q(\theta)-q(\theta+t)} = M(t)$$

$$\frac{d}{dt}M(t) = \frac{-dq(\theta+t)}{dt}\,e^{q(\theta)-q(\theta+t)}$$

$$\left.\frac{d}{dt}M(t)\right|_{t=0} = \frac{-dq(\theta+t)}{dt}\,e^{q(\theta)-q(\theta)}$$

$$= -dq(\theta+t)/dt = \theta = E[Y]$$

$$\Rightarrow q(\theta+t) = -\theta[t+\theta] + C(\theta) \ni C = \text{CONST WRT. } t$$

$$\therefore q(\theta) = -\theta^2 + C(\theta)$$

THUS

$$M(t) = e^{[-\cancel{\theta^2}+\cancel{C}]-[-\theta(t+\cancel{\theta})+\cancel{C}]}$$

$$= e^{\theta t} \Leftarrow \text{MGF FOR } n(\theta, 1)$$

[MGF FOR $n(\mu, \sigma^2) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

(7-34) $X_i \sim e^{p(\theta)K(x)+S(x)+q(\theta)}$; $a<x<b$

$$\begin{cases} Y_1 = \sum^n K(X_i) \\ Y_i = X_i \quad , i=2,\dots,n \end{cases}$$

$$\begin{cases} X_i = Y_i \quad ; i=2,\dots,n \\ X_1 = r(Y_1,\dots Y_n) \end{cases}$$

$\frac{dX_i}{\delta Y_i} = 1 \quad ; i=2,\dots,n$

$\frac{\delta X_1}{\delta Y_i} = \frac{\delta r}{\delta Y_i}$

$$\begin{bmatrix} \frac{\delta r}{\delta Y_1} & 1 & 1 \\ \frac{\delta r}{\delta Y} & 1 & 1 \\ \vdots & 1 & 1 \end{bmatrix}$$

THEN, LET $|J| = T(Y_1,\dots,Y_n)$

$\Rightarrow \cancel{g(Y_1,Y_2,\dots,Y_n) = |J|\, e^{p(\theta)K(Y_1)+S(Y_1)+q(\theta)} \prod_{i=2}^{n} e^{p(\theta)K(Y_i)+S(Y_i)+q(\theta)}}$

NOW $f(x_1\dots x_n) = \prod_{i=1}^{n} e^{p(\theta)K(x_i)+S(x_i)+q(\theta)}$

$= e^{nq(\theta)} \prod_{i=1}^{n} e^{p(\theta)K(x_i)+S(x_i)}$

$g(Y_1\dots Y_n) = T(Y_1\dots Y_n)\, e^{nq(\theta)}\, e^{p(\theta)Y_1+S[K(Y_1)]} \times \prod_{i=2}^{n} e^{p(\theta)K(Y_i)+S(Y_i)}$

$n-1$ INTEGRATIONS EAT UP ALL Y_i, $i=2,\dots,n$.
ABSORB $e^{S[K(Y_1)]}$ INTO WHAT'S LEFT OF $T(Y_1\dots Y_n)$ AND CALL IT $R(Y_1)$

$\Rightarrow g(Y_1) = R(Y_1)\, e^{nq(\theta)+p(\theta)Y_1}$

(7-36) $X_i \sim b(1;\theta)$

$\Rightarrow \quad Y = \sum^{n} X_i \sim b(n;\theta)$

$\text{var } Y = n\theta(1-\theta)$

NOW $E\left[\frac{Y}{n}\right] = \theta$

$\bar{X} = Y/n$ IS BEST STATISTIC FOR θ

WE GOTTA FIND $\psi(\bar{X})$ (OR $\phi(Y)$)

SUCH THAT $E[\phi(Y)] = n\theta(1-\theta)$

TRY

$$\phi(Y) = n \frac{Y}{n}\left(1 - \frac{Y}{n}\right)$$

$$E[\phi(Y)] = n\, E\left[\frac{Y}{n}\left(1-\frac{Y}{n}\right)\right] = \cancel{n}(n-1)\frac{\theta(1-\theta)}{\cancel{n}}$$ (FROM TOP OF p. 235)

$$= (n-1)\,\theta(1-\theta)$$

$$\Rightarrow E\left[\frac{n\phi(Y)}{n-1}\right] = n\theta(1-\theta)$$

$$\hat{\phi}(Y) = \frac{n}{n-1}\, n \frac{Y}{\cancel{n}}\left(1 - \frac{Y}{n}\right)$$

$$= \frac{Yn^2}{(n-1)}\left(1-\frac{Y}{n}\right)$$

$$= \frac{nY(n-Y)}{n-1} \Leftarrow \text{BEST}$$

(7-37) $X_i \sim n(0,\theta)$

$\Rightarrow Y = \sum X_i^2 \Leftarrow$ SUFF

$\frac{Y}{\theta} = \frac{\sum X_i^2}{\sigma^2} \sim \chi^2_n(Y)$

$E\left[\frac{Y}{\theta}\right] = n$

$Var\left[\frac{Y}{\theta}\right] = 2n$

BUT $Var\left(\frac{Y}{\theta}\right) = \frac{1}{\theta^2} Var(Y) \Rightarrow Var(Y) = 2n\theta^2$

$\Rightarrow Var\, Y$

$= E[Y^2] - (E[Y])^2 = 2n\theta^2$

$E^2[Y] = n^2\theta^2 \Rightarrow E[Y^2] = 2n\theta^2 + n^2\theta^2$

$= n(n+2)\theta^2$

$\therefore \phi(Y) = \frac{Y^2}{n(n+2)}$ IS BEST.

(7-40)

$$f(x_1, x_2 \ldots x_n) = \prod_{i=1}^{n} e^{\sum^m p_j(\theta_1 \ldots \theta_m) K_j(x_i) + S(x) + q(\theta_1 \ldots \theta_m)}$$

$$= e^{nq(\theta_1 \ldots \theta_m)} \prod^n e^{\sum^m p_j(\theta_1 \ldots \theta_m) K_j(x_i) + S(x_i)}$$

LET $\quad Y_i = \sum^n K(X_i) \quad ; \quad i = 1, 2, \ldots, m \quad \left(\text{SINCE } n > m\right)$

$\qquad Y_i = X_i \quad ; \quad i = m+1, \ldots n$

DEFINE $\quad |J| = T(Y_1 \ldots Y_n)$; IE 1 TO 1 IF K IS MONOTONIC. LETS ASSUME MORE GENERAL. (p.143).

ie $\quad |J_i| = T_i(Y_1 \ldots Y_n)$

THEN:

$$g(Y_1 \ldots Y_n) = \sum_{i=1}^{k} T_i(Y_1 \ldots Y_n) e^{nq(\theta_1 \ldots \theta_m)} \ldots$$

$$\prod^n e^{\sum_j^m p_j(\theta_1 \ldots \theta_m) K_j[Y_{ij}]} \ldots$$

LOOKS LIKE WE GOTTA ASSUME 1 TO 1 :

$$g(Y_1 \ldots Y_n) = T(Y_1 \ldots Y_n) e^{nq(\theta_1 \ldots \theta_m)} \prod_{i=1}^{m} e^{\sum_j^m p_j(\theta_1 \ldots \theta_m) Y_j + S[Y_j]} \prod_{i=1+m}^{n} e^{\sum^m p_j(\theta_1 \ldots \theta_m) K_j[Y_j] + S[Y_j]}$$

INTEGRATING OUT ALL Y_j, $j = m+1$ TO n, AND ABSORBING $e^{S(Y_j)}$'s $\quad j = 1, \ldots m$ INTO WHAT'S LEFT OF $T(Y_1 \ldots Y_n)$ AND CALLING IT $R(Y_1 \ldots Y_m)$ LEAVES

$$g(Y_1 \ldots Y_m) = R(Y_1 \ldots Y_m) e^{nq(\theta_1 \ldots \theta_m)} \prod^m e^{\sum_{j=1}^m p_j(\theta_1 \ldots \theta_m) Y_j}$$

$$= R(Y_1 \ldots Y_m) e^{nq(\theta_1 \ldots \theta_m) + \sum_{j=1}^m p_j(\theta_1 \ldots \theta_m) Y_j}$$

(7-43)

$$f(x;\theta_1,\theta_2) = e^{p_1(\theta_1,\theta_2)K_1(x) + p_2(\theta_1,\theta_2)K_2(x) + S(x) + q(\theta_1,\theta_2)}$$

$$K_1'(x) = c\,K_2'(x)$$

$$K_1(x) = c\,K_2(x) + c_2(\theta_1,\theta_2)$$

$$\therefore\ p_1(\theta_1,\theta_2)K_1(x) = c\,p_1(\theta_1,\theta_2)K_2(x) + c_3(\theta_1,\theta_2)$$

AND

$$p_1(\theta_1,\theta_2)K_1(x) + p_2(\theta_1,\theta_2)K_2(x)$$
$$= c\,p_1(\theta_1,\theta_2)K_2(x) + p_2(\theta_1,\theta_2)K_2(x) + c_3(\theta_1,\theta_2)$$
$$= \underbrace{[c\,p_1(\theta_1,\theta_2) + p_1(\theta_1,\theta_2]}_{p(\theta_1,\theta_2)} K_2(x) + c_3(\theta_1,\theta_2)$$

LET $q_1(\theta_1,\theta_2) = c_3(\theta_1,\theta_2) + q(\theta_1,\theta_2)$

THEN

$$f(x,\theta_1,\theta_2) = e^{p(\theta_1,\theta_2)K(x) + S(x) + q_1(\theta_1,\theta_2)} \quad ; a < x < b$$

(7-44) $X_i \sim f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$; $0<x, \theta<\infty$

$Z = \frac{X_1}{X_1+X_2}$

$M_Z(t) = E\left[e^{\frac{X_1}{X_1+X_2}t}\right]$

$= \frac{1}{\theta^2}\int_0^\infty\int_0^\infty e^{\frac{x_1}{x_1+x_2}t}\, e^{-(x_1+x_2)/\theta}\, dx_1\, dx_2$

$\hat{x}_1 = \frac{x_1}{\theta} \Rightarrow dx_1 = \theta\, d\hat{x}_1$

$\hat{x}_2 = \frac{x_2}{\theta} \Rightarrow dx_2 = \theta\, d\hat{x}_2$

$\Rightarrow M_Z(t) = \int_0^\infty\int_0^\infty e^{\frac{\theta \hat{x}_1}{\theta(\hat{x}_1+\hat{x}_2)}}\, e^{-(\hat{x}_1+\hat{x}_2)}\, d\hat{x}_1\, d\hat{x}_2$

$= \int_0^\infty\int_0^\infty e^{\frac{\hat{x}_1}{\hat{x}_1+\hat{x}_2}}\, e^{-(\hat{x}_1+\hat{x}_2)}\, d\hat{x}_1\, d\hat{x}_2$

NO θ DEPENDENCE. $\therefore$ $Y = X_1 + X_2$ (COM & SUFF FOR θ) IS STAT. IND. OF $Z = X_1/X_1+X_2$

(7-46.) $X_i \sim n(\theta, \sigma^2) \qquad -\infty < \theta < \infty$

$Y = \sum^n X_i$ IS COMP. & SUFF

$Z = \sum^n a_i x_i$

$$M_Z(t) = E\left[e^{\sum^n a_i x_i}\right]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{\sum^n a_i x_i} \left[\frac{1}{\sqrt{2\pi}\,\sigma}\right]^n \prod_{i=1}^{n} e^{\frac{-(x_i - \mu)^2}{2\sigma^2}} \, dx_1 \cdots dx_n$$

LET $\hat{x}_i = x_i - \mu \Rightarrow x_i = \hat{x}_i + \mu$

$$M_Z(t) = \left[\frac{1}{\sqrt{2\pi}\,\sigma}\right]^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{\sum^n a_i (x_i + \mu)} \prod_{i=1}^{n} e^{-x_i^2/2\sigma^2} \, dx_1 \cdots dx_n$$

NOW $e^{\sum^n a_i (x_i - \mu)} = e^{\sum^n a_i x_i} \, e^{-\mu \sum a_i}$

$= e^{\sum^n a_i x_i}$ FOR $\sum^n a_i = 0$

THUS, $M_Z(t)$ IS μ INDEPENDENT AND Z IS INDEPENDENT OF $Y = \sum^n X_i$

(7-49) $\frac{X}{Y}$ & Y ARE IND.

JOINT DISTRIBUTION:

$$f\left(Y, \frac{X}{Y}\right) = f_1(Y)\, f_2\left(\frac{X}{Y}\right)$$

NOW

$$E[X^K] = E\left[Y^K \left(\frac{X}{Y}\right)^K\right]$$

$$= \int Y^K f_1(Y)\, dY \int \left(\frac{X}{Y}\right)^K f_2\left(\frac{X}{Y}\right) d\left(\frac{X}{Y}\right)$$

$$= E[Y^K]\, E\left[(X/Y)^K\right]$$

$$\therefore \quad E\left[(X/Y)^K\right] = E[X^K] \big/ E[Y^K]$$

(7-51) $X_i \sim e^{-x} \quad i = 1, \dots 5$

$X_i \sim \frac{1}{\theta} e^{-x/\theta} \overset{\theta=1}{=}$ GAMMA DIST. $\alpha = 1, \beta = 1$

$Y = \sum^5 X_i$ = SUFF AND COMPLETE FOR θ

(SEE 7.10 p222)

IF Y AND $\frac{X_1+X_2}{Y}$ ARE IND $\forall\, \theta \in (0, \infty)$,

THEN Y & $Z = \frac{X_1+X_2}{Y}$ " " FOR $\theta = 1$

$$M_Z(t) = E\left[e^{\frac{X_1+X_2}{X_1+X_2+X_3+X_4+X_5} t}\right]$$

$$= \frac{1}{\theta^n} \int_0^\infty \cdots \int_0^\infty e^{\frac{(x_1+x_2)t}{\Sigma x_i}} \prod_{i=1}^{5} e^{x_i/\theta} dx_1\, dx_2 \cdots dx_5$$

$$= \frac{1}{\theta^n} \int_0^\infty \cdots \int_0^\infty e^{\frac{(x_1+x_2)t}{\Sigma x_i}} e^{\frac{1}{\theta}\sum^5 x_i} dx_1 \cdots dx_5$$

$$= \frac{1}{\theta^n} \int_0^\infty \cdots \int_0^\infty e^{\frac{(x_1+x_2)t}{\Sigma x_i} + \frac{\Sigma x_i}{\theta}} dx_1 \cdots dx_5$$

$$\hat{x}_i = \frac{x_i}{\theta} \Rightarrow dx_i = \theta\, d\hat{x}_i$$

$$M_Z(t) = \int_0^\infty \cdots \int_0^\infty e^{\frac{\theta(\hat{x}_1+\hat{x}_2)t}{\theta \Sigma \hat{x}_i} + \Sigma \hat{x}_i} d\hat{x}_1 \cdots d\hat{x}_5$$

$$= \int_0^\infty \int_0^\infty e^{\frac{(\hat{x}_1+\hat{x}_2)t}{\Sigma \hat{x}_i} + \Sigma \hat{x}_i} d\hat{x}_1 \cdots d\hat{x}_5$$

$M_Z(t)$ IND OF θ

$\Rightarrow Z$ IS IND OF Y FOR $\theta = 1$

BOB MARKS

DUE 3/1/77

9

(8-1) $x_i \sim n(\theta, \sigma^2)$

$$\Rightarrow f(x;\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\theta)^2}{2\sigma^2}}$$

$$\ln f(x;\theta) = \ln \frac{1}{\sqrt{2\pi\sigma^2}} - \frac{(x-\theta)^2}{2\sigma^2}$$

$$\frac{\delta}{\delta\theta} \ln f(x;\theta) = \frac{(x-\theta)}{\sigma^2}$$

$$-n\frac{\delta^2}{\delta\theta^2} \ln f(x;\theta) = \frac{n}{\sigma^2}$$

C-R LWR BOUND IS

$$\sigma_Y^2 \geq \sigma^2/n$$

LET $Y = \bar{X} \Rightarrow \bar{X} \sim n(\theta, \sigma^2/n)$

$$\operatorname{Var} \bar{X} = \sigma^2/n$$

$$E[\bar{X}] = \theta$$

THUS, $\bar{X}$ MEETS CRAMER-RAO BOUND

(8-2) $X_i \sim b(1,\theta) = \binom{1}{x} \theta^{x} (1-\theta)^{1-x}$

$$f(x;\theta) = \binom{1}{x}\theta^{x}(1-\theta)^{1-x}$$

$$\ln f(x;\theta) = \ln\binom{1}{x} + x\ln\theta + (1-x)\ln(1-\theta)$$

$$\frac{\partial}{\partial\theta}\ln f(x;\theta) = \frac{x}{\theta} + \left[\frac{-(1-x)}{(1-\theta)}\right]$$

$$= \frac{x}{\theta} + \frac{x-1}{1-\theta}$$

$$\cancel{\frac{\partial^2}{\partial\theta^2} f(x;\theta) = -\frac{x}{\theta^2} - \frac{1}{(1-\theta)^2}}$$

$$\left[\frac{\partial}{\partial\theta}\ln f(x;\theta)\right]^2 = \left(\frac{x}{\theta} + \frac{x-1}{1-\theta}\right)^2$$

$$= \frac{x^2}{\theta^2} + \frac{x(x-1)}{\theta(1-\theta)} + \frac{(x-1)^2}{(1-\theta)^2}$$

$$= \left[\frac{x(1-\theta) + (\theta)(x-1)}{\theta(1-\theta)}\right]^2$$

$$= \left[\frac{x - x\cancel{\theta} + x\cancel{\theta} - \theta}{\theta(1-\theta)}\right]^2$$

$$= \left[\frac{x-\theta}{\theta(1-\theta)}\right]^2$$

$$E\left[\left(\frac{\partial}{\partial\theta} f(x;\theta)\right)^2\right] = \frac{E\left[(x-\theta)^2\right]}{\theta^2(1-\theta)^2}$$

$$= \frac{\theta(1-\theta)}{\theta^2(1-\theta)^2}$$

$$= \frac{1}{\theta(1-\theta)}$$

$$\frac{1}{n\,E\left[(\quad)^2\right]} = \frac{1}{n}\theta(1-\theta)$$

NOW: $Y = \sum X_i \sim b(n,\theta)$

$$\Rightarrow \operatorname{Var} Y = n\theta(1-\theta)$$

$$\operatorname{Var}\bar{X} = \operatorname{Var}\frac{Y}{n} = \frac{1}{n^2}\operatorname{Var} Y$$

$$= \frac{1}{n}\theta(1-\theta)$$

(CRAMER-RAO BOUND MET $\Rightarrow \bar{X}$ IS EFFICIENT

SINCE $E[\bar{X}] = \theta$

(8.3) $f(x;\theta) = \frac{1}{\theta} \quad ; 0 < x < \theta$

$$\ln f(x;\theta) = \ln \frac{1}{\theta}$$
$$= -\ln \theta \quad ; 0 < x < \theta$$
$$\frac{\delta}{\delta\theta} \ln f(x;\theta) = \frac{-1}{\theta} \quad ; 0 < x < \theta$$
$$\left[\frac{\delta}{\delta\theta} \ln f(x;\theta)\right]^2 = 1/\theta^2 \quad ; 0 < x < \theta$$
$$E\left[\left\{\frac{\delta}{\delta\theta} \ln f(x;\theta)\right\}^2\right] = E\left[1/\theta^2\right] = 1/\theta^2$$
$$\Rightarrow \frac{1}{nE\left[\left\{\frac{\delta}{\delta\theta} \ln f(x;\theta)\right\}^2\right]} = \theta^2/n$$

NOW

LET $Y_n = Max\ X_i$

$$F_n(Y_n) = F^n(Y_n)$$
$$f_n(y_n) = n\, f(Y_n)\, F^{n-1}(Y_n)$$
$$= n \frac{1}{\theta} \left(\frac{Y_n}{\theta}\right)^{n-1}$$
$$= \frac{n}{\theta^n} Y_n^{n-1}$$
$$E[Y_n] = \frac{n}{\theta^n} \int_0^\theta Y_n^n\, dY = \frac{n}{\theta^n} \frac{1}{n+1} Y_n^{n+1} \Big|_0^\theta$$
$$= \frac{n\theta}{n+1}$$
$$E[Y_n^2] = \frac{n}{\theta^n} \int_0^\theta Y_n^{n+1}\, dY_n = \frac{n}{\theta^n(n+2)} Y^{n+2} \Big|_0^\theta$$
$$= \frac{n\theta^2}{n+2}$$
$$Var\, Y_n = \left(\frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2\right)\theta^2$$
$$= \cancel{\frac{n^2+n-n^2-2n}{(n+2)(n+1)} \theta^2 = \frac{-n}{(n+2)(n+1)} \theta^2}$$
$$= \frac{n(n+1)^2 - n^2(n+2)}{(n+1)^2(n+2)} \theta^2 = \frac{n^3+2n^2+n-n^3-2n^2}{(n+1)^2(n+2)} \theta^2$$
$$= \frac{n}{(n+1)^2(n+2)} \theta^2$$
$$\Rightarrow E\left[\frac{(n+1)Y_n}{n}\right] = \left(\frac{n+1}{n}\right)\left(\frac{n}{n+1}\right)\theta = \theta$$

$$Var\left[\frac{(n+1)Y_n}{n}\right] = \frac{(n+1)^2}{n^2} Var\, Y_n$$
$$= \frac{\cancel{(n+1)^2}}{n^{\cancel{2}}} \frac{\cancel{n}}{\cancel{(n+1)^2}(n+2)} \theta^2$$
$$= \frac{1}{n(n+2)} \theta^2$$

THOUGH $\frac{(n+1)Y_n}{n}$ IS UNBIASED, IT IS NOT EFFICIENT.

Is $\sigma^2_{\frac{n+1}{n}Y_n} = \frac{\theta^2}{n(n+2)} \geq \frac{\theta^2}{n}$?

(8-4) $f(x;\theta) = \dfrac{1}{\pi[1+(x-\theta)^2]}$

$$\ln f(x;\theta) = \ln \frac{1}{\pi} + \ln \frac{1}{1+(x-\theta)^2}$$

$$\frac{\delta}{\delta\theta} \ln f(x;\theta) = -\frac{\delta}{\delta\theta} \ln [1+(x-\theta)^2]$$

$$= \frac{-2(x-\theta)}{1+(x-\theta)^2}$$

$$\left[\frac{\delta}{\delta\theta} \ln f(x;\theta)\right]^2 = \frac{4(x-\theta)^2}{[1+(x-\theta)^2]^2}$$

$$E\left[\left(\frac{\delta}{\delta\theta} \ln f(x;\theta)\right)^2\right] = \frac{4}{\pi}\int_{-\infty}^{\infty} \frac{(\hat{x}-\theta)^2 \, d\hat{x}}{[1+(\hat{x}-\theta)^2]^3}$$

$$x = \hat{x} - \theta$$

$$E[(\quad)^2] = \frac{4}{\pi}\int_{-\infty}^{\infty} \frac{x^2 \, dx}{(1+x^2)^3}$$

$$= 8/\pi \int_0^{\infty} \frac{x^2}{(1+x^2)^3} dx$$

$$= 8/\pi \times \frac{\pi}{16} = 1/2$$

C.R. BOUND IS THUS

$$\frac{1}{nE[(\quad)^2]} = \frac{1}{n/2} = 2/n$$

(8-5)

$$\int f(x;\theta)\,dx = 1$$

$$\int \frac{\delta}{\delta\theta} f(x;\theta)\,dx = 0$$

$$= \int f(x;\theta) \frac{\delta}{\delta\theta} \ln f(x;\theta)$$

DIFFERENTIATE AGAIN:

$$0 = \int \frac{\delta}{\delta\theta} f(x;\theta) \frac{\delta}{\delta\theta} \ln f(x;\theta)$$

$$+ f(x;\theta) \frac{\delta^2}{\delta\theta^2} \ln f(x;\theta)$$

$$= \int f(x;\theta) \left[\frac{\delta}{\delta\theta} \ln f(x;\theta)\right]^2$$

$$+ \int f(x;\theta) \frac{\delta^2}{\delta\theta^2} \ln f(x;\theta)$$

$$= E\left[\left(\frac{\delta}{\delta\theta} \ln f(x;\theta)\right)^2\right] + E\left[\frac{\delta^2}{\delta\theta^2} \ln f(x;\theta)\right] = 0$$

OR

$$E\left[\left(\frac{\delta}{\delta\theta} \ln f(x;\theta)\right)^2\right] = -E\left[\frac{\delta^2}{\delta\theta^2} \ln f(x;\theta)\right]$$

(8-6)

(a) POISSON: $X_i \sim \dfrac{\theta^x e^{-\theta}}{x!}$

$m_1 = \frac{1}{n}\sum^n X_i = \bar{X}$

$E[\bar{X}] = \theta$

$\bar{X}$ IS BEST (& THUS CONSISTANT) STATISTIC FOR θ

(b) $X_i\ n(\theta_1, \theta_2)$

$m_1 = \frac{1}{n}\sum_1^n X_i = \bar{X}$

$\theta_1 = E[\bar{X}] = E[m_1]$ IS BEST (& THUS CONSIST) STATISTIC FOR θ_1.

$m_2 = \frac{1}{n}\sum^n X_i^2 =$

$S^2 = m_2 - m_1^2$ IS CONSISTANT STAT FOR $\sigma^2 = \theta_2$

(c) ~~$f(x;\theta) = \theta x^{\theta-1}$; $0 < x < 1$, $\theta > 0$~~

~~$m_1 = \frac{1}{n}\sum_n X_i$~~

~~$M(t) = E[e^{tx}] = \theta\int_0^1 x^{\theta-1} e^{-tx}\,dx$~~ (ARG)

~~$E[x] = \theta\int_0^1 x^\theta dx = \frac{\theta}{1+\theta} x^{\theta+1}\Big|_0^1 = \frac{\theta}{\theta+1}$~~

$\Rightarrow E[\bar{X}] = \frac{\theta}{\theta+1}$

$Y = \sum^n x$

~~$f(Y) = \prod^n f(x_i)$~~

~~$= \theta^n \prod^n x_i^{\theta-1} = \theta^n \left(\prod^n x_i\right)^{\theta-1}$~~

~~$f(\bar{x}) = \frac{1}{n} f\left(\frac{y}{n}\right) = \frac{1}{n}\theta^n\left(\prod^n\ \right)^{\theta-1}$~~

$$f(x;\theta) = \theta\, x^{\theta-1}$$

LET $Y = -\theta \ln x \Rightarrow x = e^{-Y/\theta}$ $\quad |J| = +\frac{1}{\theta} e^{-Y/\theta}$

$$f(Y;\theta) = \left[\tfrac{1}{\theta} e^{-Y/\theta}\right] \theta \left[e^{-Y/\theta}\right]^{\theta-1}$$
$$= e^{-Y/\theta}\, e^{-Y}\, e^{Y/\theta}$$
$$= e^{-Y} \quad ; 0 < Y < \infty$$

$$E[Y] = 1 = E[-\theta \ln x]$$
$$\Rightarrow E[\ln x] = -1/\theta$$

LET $Y = -\frac{1}{\theta} \ln x \qquad x = e^{-Y\theta}$

$$f(x;\theta) = e^{\ln\theta + (\theta-1)\ln x}$$

$\sum^{n} \ln x$ IS SUFF

$= \ln \prod^{n} x$ is SUFF

$\ln \prod^{n}$

$$E[X] = \theta \int_0^1 x^{\theta}\, dx = \frac{\theta}{\theta+1}$$
$$\Rightarrow E[\bar{X}] = \frac{\theta}{\theta+1} \quad \ldots$$

(CAN'T CRACK IT)

(8-7) WE ONLY NEED TO SHOW THAT Y_n CONVERGES STOCHASTICALLY TO θ. SINCE $\sigma_Y \to 0$, WE MAY CONCLUDE (FROM p. 53 1.84), THAT THE DISTRIBUTION OF Y APPROACHES A DEGENERATE CASE. THE DEF. OF STOCHASTIC CONVERGENCE (p. 175) IS A LIMITING DEGENERATE pdf. THUS, BY DEF. 3 ON 253, Y_n IS A CONSISTANT STATISTIC

(8-8)

(a) $f(x;\theta) = \theta^x e^{-\theta}/x!$ $x = 0, 1, 2, \ldots$ $0<\theta<\infty$

$$L(x_1 \ldots x_n;\theta) = \prod^n \theta^{x_i} e^{-\theta}/x!$$
$$= \theta^{\sum^n x_i} e^{-n\theta} \Big/ \prod^n x_i!$$

$$\ln L(x_1 \ldots x_n;\theta) = \sum^n x_i \ln\theta - n\theta$$
$$= n\bar{x}\ln\theta - n\theta$$

$$\frac{\delta}{\delta\theta}\ln L(x_1 \ldots x_n;\theta) = \frac{n\bar{x}}{\theta} - n = 0 = \frac{\bar{x}}{\theta} - 1$$
$$\Rightarrow \bar{X} = \hat{\theta}$$

($\bar{x}$ IS SUFF FOR θ)

(b) $f(x;\theta) = \theta x^{\theta-1}$; $0<x<1$, $0<\theta<\infty$

$$L(x_1 \ldots x_n;\theta) = \theta^n \prod^n x_i^{\theta-1}$$
$$= \theta^n \left(\prod^n x_i\right)^{\theta-1}$$

$$\ln L = n\ln\theta + (\theta-1)\ln\prod^n x_i$$

$$\frac{\delta}{\delta\theta}\ln L = \frac{n}{\theta} - \ln\prod^n x_i$$
$$\Rightarrow \frac{n}{\theta} = \ln\prod^n x_i$$
$$\frac{\theta}{n} = \frac{1}{\ln\prod^n x_i}$$
$$\hat{\theta} = -n\Big/\ln\prod^n x_i$$

$\prod^n X_i$ IS SUFF SINCE $f(x;\theta)$ IS A BETA DISTRIBUTION (SEE 7.11 ON p. 223)

(c) $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$; $0<x$; $\theta<\infty$

$$L(x_1 \ldots x_n;\theta) = \frac{1}{\theta^n} e^{-\sum^n x_i/\theta}$$
$$= \theta^{-n} e^{-n\bar{x}/\theta}$$

$$\ln L = -n\ln\theta - n\bar{x}/\theta = 0$$

$$\frac{\delta}{\delta\theta}\ln L = 0 = \frac{-n}{\cancel{\theta}} + n\bar{x}/\theta^{\cancel{2}} = -1 + \bar{x}/\theta$$
$$\Rightarrow 1 = \bar{x}/\theta \Rightarrow \hat{\theta} = \bar{X}$$

$Y = n\bar{X} = \sum^n X_i$ IS SUFFICIENT SINCE $f(x;\theta)$ IS AN ONE PARAMETER EXPON. FAMILY WITH $K(x) = x$

(d) $f(x;\theta) = \frac{1}{2} e^{-|x-\theta|}$; $-\infty \leq x;\theta < \infty$

$L(x_1 \ldots x_n;\theta) = \frac{1}{2^n} e^{-\sum^n |x_i-\theta|}$

$\ln L = \ln \frac{1}{2^n} - \sum^n |x_i-\theta|$

TO MAXIMIZE $\ln L$, WE WISH TO MAKE $\sum^n |x_i-\theta|$ AS SMALL AS POSSIBLE. IT WAS SHOWN IN 1.71 (p.45), THAT $E(|x_i-\theta|)$ IS MINIMUM FOR θ = MEDIAN. SUMMING n TERMS GIVES US

$$\hat{\theta} = n\, m$$

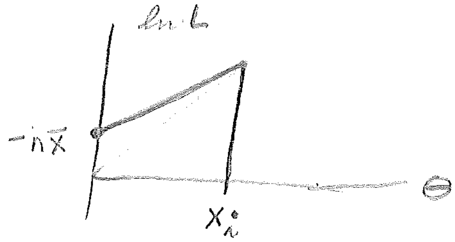
WHERE m = MEDIAN OF THE SAMPLE. (NOT SUFFICIENT IF MY INTUITION IS CORRECT)

(e) $f(x;\theta) = e^{-(x-\theta)}$; $\theta \leq x < \infty$, $-\infty < \theta < \infty$

$L(x_1 \ldots x_n;\theta) = e^{-\sum^n (x_i-\theta)}$; $\theta \leq x_i \leq \infty$

$= e^{-\sum^n x_i + n\theta}$

$\ln L = -\sum^n X_i + n\theta$; $\theta \leq x_i < \infty$



$\ln L$ ACHEIVES IT'S MAXIMUM WHEN X_i IS MAX, <u>BUT</u>, WE KNOW $\theta \leq X_{(1)} \ni X_{(1)}$ IS FIRST ORDER STATISTIC. SINCE $X_{(1)}$ IS THE LARGEST POSSIBLE VALUE θ CAN HAVE, AND SINCE $\ln L$ IS INCREASING,

$$\hat{\theta} = X_{(1)}$$

(f) $f(x;\theta) = \theta^{x}(1-\theta)^{1-x}, \quad x = 0,1 \quad 0 \le \theta \le 1$

$$L(x_1, \ldots x_n;\theta) = \theta^{\Sigma x}(1-\theta)^{\Sigma(1-x)}$$

$$= \theta^{\Sigma x}(1-\theta)^{n-\Sigma x}$$

$$\ln L = \Sigma x \ln\theta + (n - \Sigma x)\ln(1-\theta)$$

$$\frac{d}{d\theta}\ln L = \Sigma x/\theta - n+\Sigma x/(1-\theta) = 0$$

$$= (1-\theta)\Sigma x - \theta(n + \Sigma x)$$

$$= \Sigma x - \cancel{\theta\Sigma x} - n\theta + \cancel{\theta\Sigma x}$$

$$\Rightarrow \theta = \Sigma x/n = \bar{x}$$

$$f(x;\theta) = e^{x\ln\theta + (1-x)\ln(1-\theta)}$$

$$e^{x[\ln\theta - \ln(1-\theta)] + \ln(1-\theta)}$$

$K(x) = \Sigma x$ IS SUFFICIENT

(8-9) $X_i \sim \frac{1}{\theta_2} e^{-(x-\theta_1)/\theta_2}$; $\theta_1 < x < \infty$

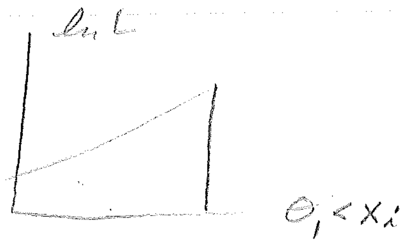
$-\infty < \theta_1 < \infty$

$0 < \theta_2 < \infty$

$$L(x_1, \ldots, x_n; \theta) = \frac{1}{\theta_2^n} e^{-\sum_n \left(\frac{x-\theta_1}{\theta_2}\right)}$$

$$= \frac{1}{\theta_2^n} e^{-\frac{\sum x_i}{\theta_2} + \frac{n\theta_1}{\theta_2}}$$

$$\ln L = -n \ln \theta_2 - \frac{\sum X_i}{\theta_2} + \frac{n\theta_1}{\theta_2}$$



$\therefore$ CHOOSE $\hat{\theta}_1$ TO BE $\overset{\text{min}}{\cancel{\text{max}}}\ X_i = Y_1$

$$\Rightarrow \ln L = -n \ln \theta_2 - \frac{\sum x_i}{\theta_2} + nY_1/\theta_2$$

$$= -n \ln \theta_2 - \sum (x_i - Y_1)/\theta_2$$

$$\frac{\delta \ln L}{\delta \theta} = \frac{-n}{\cancel{\theta}_2} + \sum (x_i - Y_1)/\theta_2^{\cancel{2}} = 0$$

$$\frac{1}{n} \sum (x_i - Y_1) = \hat{\theta}_2$$

(8-10) $X_i \sim f(x;\theta) \quad ; \theta \in \Omega$

$L(\theta) = \prod^n f(x_i;\theta)$

$Y = u(X_1 \ldots X_n)$ IS SUFF.

$Y \sim g(y;\theta_0)$

PROVE $\quad g(y;\theta) = g(y;\theta_0)\,\dfrac{L(\theta)}{L(\theta_0)}$

OR $\quad \dfrac{g(y;\theta)}{L(\theta)} = \dfrac{g(y;\theta_0)}{L(\theta_0)}$ (1)

$\cancel{g(y;\theta) = |J| \prod^n f[u_i}$ ...

BY FAC. THM (p. 217) Y IS SUFF IF

$L(\theta) = \prod^n f(x_i;\theta) = g(y;\theta)\,H(x_1 \ldots x_n)$ (2)

THUS $\Rightarrow L(\theta_0) = \prod^n f(x_i;\theta_0) = g(y;\theta_0)\,H(x_1 \ldots x_n)$ (3)

(1) IMMEDIATELY FOLLOWS FROM (2) & (3)

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$\theta \in \{\theta = 1, 2\}$

(9-1) $f(x;\theta) = \theta x^{\theta-1}$; $0 < x < 1$

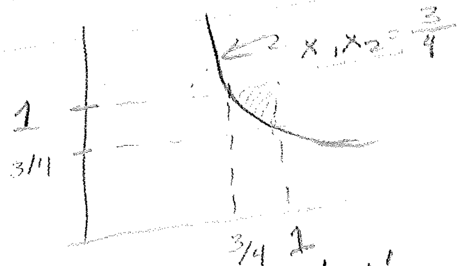
$H_0: \theta = 1$

$H_1: \theta = 2$

C = CRITICAL REGION = $\{(x_1, x_2) ; \frac{3}{4x_1} \le x_2\}$

(IF IN HERE, REJECT H_0)

$K(\theta) = Pr[\text{rej } H_0]$



$$K(\theta) = \int_{3/4}^{1}\int_{3/4x_2}^{1} \theta^2 x_1^{\theta-1} x_2^{\theta-1}\, dx_1\, dx_2$$

$$= \theta^2 \int_{3/4}^{1} \frac{x_1^{\theta}}{\theta}\Big|_{3/4x_1}^{1} dx_2 = \theta \int_{3/4}^{1} x_2^{\theta-1}\left(1 - \left(\tfrac{3}{4}\right)^{\theta} x_2^{-\theta}\right) dx_2$$

$$= \theta\left[\frac{x_2^{\theta}}{\theta} - \left(\tfrac{3}{4}\right)^{\theta} \ln x_2 \Big|_{3/4}^{1}\right]$$

$$= \theta\left[\frac{1}{\theta} - \frac{1}{\theta}\left(\tfrac{3}{4}\right)^{\theta} + \left(\tfrac{3}{4}\right)^{\theta} \ln 3/4\right]$$

$= \frac{1}{4} + \frac{3}{4} \ln 3/4$

$K(1) = 1 - 3/4 + 3/4 \ln 3/4$

$K(2) = 2\left[\frac{1}{2} - \frac{1}{2} \cdot 9/16 + \frac{9}{16} \ln 3/4\right]$

$= 1 - 9/16 + 9/8 \ln 3/4$

$= 7/16 + 9/8 \ln 3/4$

9-6 $X \sim p(\theta)$

$H_0: \theta = \frac{1}{2}$

$H_1: \theta < \frac{1}{2}$

$\Omega = \{\theta;\ 0 < \theta < \frac{1}{2}\}$

$n = 12$

REJECT H_0 IF $\sum^{n} X_i \leq 2$

$K(\theta) = P[\text{REJ } H_0 | \theta]$

$= P[\sum X_i \leq 2]$

NOW $X_i \sim p(\theta) \Rightarrow m_x(t) = e^{\theta(e^t - 1)}$

$\Rightarrow M_Y(t) = e^{n\theta(e^t - 1)}$

$\therefore Y \sim p(n\theta)$

$K(\theta) = P[Y < 2 | \theta]$

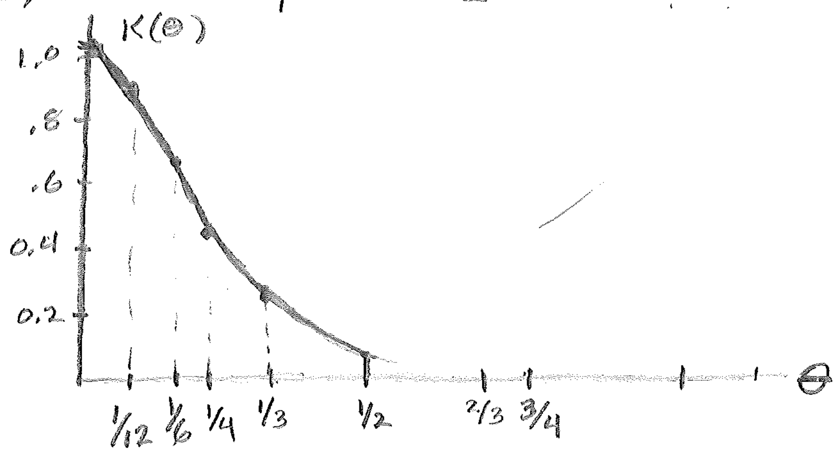
$\frac{1}{2}$: $P[Y \leq 2 | \theta = \frac{1}{2}] = P[Y \leq 2 | n\theta = 6] = 0.062$

$\frac{1}{3}$: $P_1[Y \leq 2 | n\theta = n\frac{1}{3} = 4] = 0.238$

$\frac{1}{4}$: $P_1[Y \leq 2 | n\theta = \frac{12}{4} = 3] = 0.423$

$\frac{1}{6}$: $P_1[Y \leq 2 | n\theta = 2] = 0.677$

$\frac{1}{12}$; $P[Y \leq 2 | n\theta = 1] = 0.920$



H_0 IS TRUE WHEN $\theta = \frac{1}{2}$

$\Rightarrow \alpha = $ SIG. LEVEL $= 0.062$

(9-5) $X \sim n(\theta, (5000)^2) \Rightarrow \bar{X} \sim n\left(\theta, \frac{(5000)^2}{n}\right)$

$H_0: \theta \leq 30{,}000$

$H_1: \theta > 50{,}000$ REJECT H_0 WHEN $\bar{X} \geq c$

$K(30{,}000) = 0.01$ $K(35{,}000) = 0.01$

$$\frac{\bar{X} - \theta}{5000/\sqrt{n}} \sim n(0,1)$$

$$K(\theta) = P_r[\bar{X} \geq c] = \int_{\frac{c-\theta}{5000/\sqrt{n}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

$$= 1 - \Phi\left[\frac{(c-\theta)\sqrt{n}}{5000}\right]$$

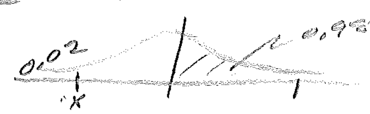
$$K(30{,}000) = 0.01 = 1 - \Phi\left[\frac{(c-30000)\sqrt{n}}{5000}\right.$$

$$\Rightarrow \Phi\left[\frac{(c-3000)\sqrt{n}}{5000}\right] = 0.98$$

$$P_g\ 400 \Rightarrow \frac{(c-3000)\sqrt{n}}{5000} = 2.05 \quad (1)$$

$$K(35{,}000) = 0.98 = 1 - \Phi\left[\frac{(c-35000)\sqrt{n}}{5000}\right]$$

$$\Rightarrow \Phi\left[\frac{(c-35000)\sqrt{n}}{5000}\right] = 0.02$$



$$\therefore \frac{(c-35000)\sqrt{n}}{5000} = -2.05 \quad (2)$$

FROM (1) & (2) $\frac{\sqrt{n}}{5000} = \frac{-2.05}{c-35{,}000} = \frac{2.05}{c-30{,}000}$

$\Rightarrow c - 35{,}000 = -c + 30000$

$2c = 65000 \Rightarrow c = 32{,}500$

$$n = \left(\frac{5000(2.05)}{2500}\right)^2 = (4.05)^2 = 16.4$$

16 OR 17 SAMPLES

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9-7, 10, 11, ~~12~~, 13, 15

(9-7) $H_0: \theta = \theta' = 0$

$H_1: \theta = \theta'' = -1$

$$\frac{L_0}{L_1} = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n}{\left(\frac{1}{\sqrt{2\pi}}\right)^n} \; \frac{e^{-\frac{1}{2}\sum^n x_i^2}}{e^{-\frac{1}{2}\sum^n (x_i+1)^2}}$$

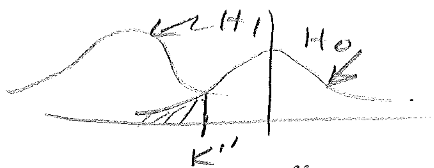
$$= \exp\left[-\tfrac{1}{2}\sum x^2 + \tfrac{1}{2}\sum x^2 + \sum x_i + \tfrac{1}{2}n\right] \le k$$

$$\Rightarrow \sum x_i \le k'$$

$$\bar{x} \le k''$$

$\bar{x}$ UNDER $H_0 \sim n(0, \frac{1}{25})$

$\alpha = 0.05 = P_r[H_1 / H_0]$



$$0.05 = \int_{-\infty}^{k''} n(0, \tfrac{1}{25})\,dx = 1 - \int_{-\infty}^{k''} n(0, \tfrac{1}{25})\,dx$$

$$\Rightarrow 0.95 = \int_{-\infty}^{k''} n(0, \tfrac{1}{25})\,dx$$

$$= \int_{-\infty}^{5k''} n(0,1)\,dx$$

$$\Rightarrow 5k'' = 1.645 \Rightarrow k'' = 0.329$$

$$\beta = PWR = P_r[H_1 / H_1]$$

$$= \int_{-\infty}^{k''} n(-1, \tfrac{1}{25})\,dx$$

(UNDER H_0, $\bar{x} \sim n(-1, \frac{1}{25})$)

$$\beta = \int_{-\infty}^{5(k'+1)} n(0,1)\,dx$$

$$= \int_{-\infty}^{6.645} n(0,1)\,dx$$

THIS IS OFF THE TABLE ON p. 400

$$\Rightarrow \beta = 0.999+$$

(9-10) $x_i \sim n(0, \sigma^2)$ $n = 10$

$\alpha = 0.05$

$H_0: \sigma_0^2 = 1$

$H_1: \sigma_1^2 = 2$

$$\frac{L_1}{L_0} = \frac{\left(\frac{1}{\sqrt{2\pi}\sigma_1}\right)^n}{\left(\frac{1}{\sqrt{2\pi}\sigma_0}\right)^n} \frac{e^{-\frac{1}{2\sigma_1^2}\Sigma x^2}}{e^{-\frac{1}{2\sigma_0^2}\Sigma x^2}} \geq k$$

$$\Rightarrow -\frac{1}{2}\underbrace{\left[\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right]}_{<0} \Sigma x^2 \geq k$$

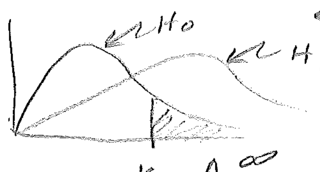
$$\Rightarrow \Sigma x^2 \geq k' = c$$

NOW $\alpha = 0.05 = Pr[H_1/H_0]$

UNDER $H_0 \Rightarrow x_i \sim n(0,1)$

$x_i^2 \sim \chi^2_{(1)}$

$\sum^{10} x_i^2 \sim \chi^2_{(n)} = \chi^2_{10}$



$0.05 = \int_k^\infty \chi^2_{10}\, dx$

OR $0.95 = \int_0^k \chi^2_{10}\, dx \Rightarrow k = 18.3$

THIS IS ALSO BEST CRITICAL REGION FOR TESTING $H_0: \sigma_2 = 1$ VS $H_1: \sigma^2 = 4$ OR $H_1: \sigma_2^2 > 1$, SINCE IN BOTH CASES, POWER IS MAXIMIZED.

(9-11) $X_i \sim \theta X^{\theta - 1}$

$H_0: \theta = 1$

$H_1: \theta = 2$

$$\frac{L_1}{L_0} = \frac{2^n \prod^n x_i}{1^n} \geq k$$

$$\prod^n x_i \geq c$$

THUS

$$C = \left\{ (x_1, \ldots x_n);\ \prod_{i=1}^{n} x_i \geq c \right\}$$

(9-12) $X_i \sim n(\theta_1, \theta_2)$

$H_0: \theta_1 = 0, \theta_2 = 1$

$H_1: \theta_1 = 1, \theta_2 = 4$

UNDER H_0: $f(x_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

UNDER H_1: $f(x_i) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2\cdot 4}(x-1)^2}$

$$\frac{L_0}{L_1} = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n}{\left(\frac{1}{2\sqrt{2\pi}}\right)^n} \frac{e^{-\frac{1}{2}\Sigma x^2}}{e^{-\frac{1}{2}\cdot\frac{1}{4}\Sigma(x-1)^2}} \le K$$

$$-\frac{1}{2}\left[\Sigma x^2 - \frac{1}{4}\Sigma(x-1)^2\right] \le K'$$

$$-\frac{1}{2}\left[\Sigma x^2 - \frac{1}{4}\Sigma x^2 + \frac{1}{2}\Sigma x - \frac{n}{4}\right] \le K'$$

$$-\frac{1}{2}\left[\frac{3}{4}\Sigma x^2 + \frac{1}{2}\Sigma x - \frac{n}{4}\right] \le K'$$

$$\frac{3}{4}\Sigma x^2 + \frac{1}{2}\Sigma x \ge K''$$

$$3\Sigma x^2 + 2\Sigma x \ge K''' = C$$

(9-13) $X_i \sim n(\theta, 100)$

$H_0 : \theta = 75 = \theta_0$

$H_1 : \theta = 78 = \theta_1$

$$\frac{L_1}{L_0} = \frac{\left(\frac{1}{\sqrt{2\pi}\,10}\right)^n}{\left(\frac{1}{\sqrt{2\pi}\,10}\right)^n} \; \frac{e^{-\frac{1}{200}\sum(x-\theta_1)^2}}{e^{-\frac{1}{200}\sum(x-\theta_0)^2}} \geq K$$

$$-\frac{1}{200}\sum\left[(x-\theta_1)^2-(x-\theta_0)^2\right] \geq k$$

$$-\frac{1}{200}\sum\left[\cancel{x^2} - 2x\theta_1+\theta_1^2 - \cancel{x^2}+2x\theta_0-\theta_0^2\right] \geq K$$

$$\sum X_i(2\theta_0-2\theta_1)+(\theta_1^2-\theta_0^2) \leq K'$$

$$\underbrace{(\theta_0-\theta_1)}_{<0}\sum x \leq K''$$

$$\Rightarrow \bar{X} \geq C$$

UNDER H_0, $\bar{X} \sim n(75, \frac{100}{n})$

$$0.05 = \alpha = P_r[H_1 / H_0] = \int_c^\infty n(75, \tfrac{100}{n})\,dx$$

$$= \int_{\frac{c-75}{10/\sqrt{n}}}^{\infty} n(0,1)\,dx = 1 - \int_{-\infty}^{\frac{c-75}{10/\sqrt{n}}} n(0,1)\,dx$$

$$\Rightarrow \int_{-\infty}^{\frac{c-75}{10/\sqrt{n}}} n(0,1)\,dx = 0.95 = \Phi\left[\frac{c-75}{10/\sqrt{n}}\right] \quad (1)$$

$$\beta = 0.90 = P_r[H_1 / H_1]$$

UNDER H_1, $\bar{X} \sim n[78, \frac{100}{n}]$

$$\beta = 0.9 = \int_c^\infty n(78, \tfrac{100}{n})\,dx = \int_{\frac{c-78}{10/\sqrt{n}}}^{\infty} n(0,1)\,dx$$

$$= \int_{-\infty}^{\frac{78-c}{10/\sqrt{n}}} n(0,1)\,dx$$

FROM (1): $\frac{c-75}{10/\sqrt{n}} = 1.645$

FROM (2): $\frac{c-78}{10/\sqrt{n}} = -1.282$

$$\frac{c-75}{c-78} = \frac{-1.645}{1.282}$$

$$c-75 = \frac{-1.645}{1.282}(c-78) \Rightarrow c\left[1+\frac{1.645}{1.282}\right] = 75 + \frac{1.645}{1.282}78$$

$$c(2.283) = 175.08 \Rightarrow c = 76.7$$

$$n = \left(\frac{c-75}{10(1.645)}\right)^{-2} = \left(\frac{1.7}{16.45}\right)^{-2} = 96 \text{ OR } 97$$

(9-15) $X_i \sim p(\theta)$ $\quad n = 10$

$\sum^{10} x \sim p(10\theta)$

$H_0 : \theta = 0.1$

$H_1 : \theta = 0.5$

$$\frac{L_0}{L_1} = \frac{\prod^{10} (0.1)^{x_i} e^{-0.1} / x_i!}{\prod^{10} (0.5)^{x_i} e^{-0.5} / x_i!}$$

$$= (.2)^{\sum x_i} e^{0.4} \le K$$

$$\sum x_i \underbrace{\ln(.2)}_{<0} \le K'$$

$$\Rightarrow \sum x_i \ge K'$$

$\alpha = Pr[H_1 | H_0]$

UNDER H_0, $\sum x_i \sim p(1) = \frac{(1)^x e^{-1}}{x!}$

$$\Rightarrow \alpha = \sum_3^{\infty} \frac{e^{-1}}{x!}$$

$$= 1 - \sum_{x=0}^{2} \frac{e^{-1}}{x!} = 1 - 0.920 = 0.08$$

$\beta = Pr[H_1 / H_1]$

UNDER H_1 $\sum x_i \sim p(5) = \frac{5^x e^{-5}}{x!}$

$$\beta = \sum_3^{\infty} 5^x e^{-5} / x!$$

$$= 1 - \sum_{x=0}^{2} \frac{5^x e^{-5}}{x!}$$

$$= 1 - .125 = 0.875$$

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(9-18) $\quad x_i \sim n(\theta, 4) \qquad n = 25$

$H_0 : \theta = 0$

$H_1 : \theta > 0$

UMP TEST IS

$$\bar{X} \geq c = \frac{3}{5}$$

$$\bar{X} \sim n\left(\theta, \frac{4}{25}\right) = \frac{1}{\sqrt{2\pi}\, 2/5} e^{-\frac{1}{2}\left(\frac{25}{4}\right)(x-\theta)^2}$$

$$K(\theta) = P_r[H_1 \mid \theta]$$

UNDER $H_1 : \bar{x} \sim n\left(\theta, \frac{4}{25}\right)$

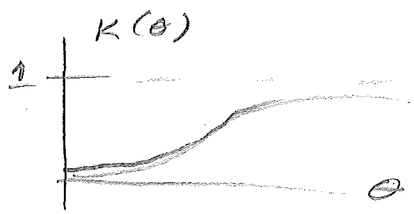
$$\Rightarrow K(\theta) = P_r\left[\bar{x} > \frac{3}{5} \,\middle|\, \theta\right]$$

$$= \frac{1}{\sqrt{2\pi}\, 2/5} \int_{3/5}^{\infty} e^{-\frac{1}{2}\left(\frac{25}{4}\right)(x-\theta)^2} dx$$

$$= 1 - \int_{-\infty}^{3/5} \frac{1}{\sqrt{2\pi}\, \frac{2}{5}} e^{-\frac{1}{2}\left(\frac{25}{4}\right)(x-\theta)^2} dx$$

$$= 1 - \int_{-\infty}^{\frac{\frac{3}{5} - \theta}{2/5}} n(0,1)\, dx$$

$$= 1 - N\left[\frac{3/5 - \theta}{2/5}\right] \qquad ; \theta \geq 0$$



(9-20) $H_0: \theta = \theta'$

$H_1: \theta < \theta' \quad (\theta = \theta'')$

$$\frac{L_0}{L_1} = \left(\frac{\theta''}{\theta'}\right)^{n/2} e^{-\left(\frac{\theta''-\theta'}{2\theta'\theta''}\right)\sum^n x_i^2} \leq K$$

$$-\frac{\theta''-\theta'}{2\theta'\theta''}\sum x_i^2 \leq K'$$

$$\theta'' < \theta' \Rightarrow \theta''-\theta' < 0$$

$$\Rightarrow \sum x_i^2 \leq K'' = C$$

UMP CRIT REG $= \{(x_1 \ldots x_n);\ \sum^n x_i^2 \leq C\}$

(9-21) $H_0: \theta = \theta'$

$H_1: \theta \neq \theta' \quad (\theta = \theta'' \neq \theta')$

THUS

$$\frac{L_0}{L_1} = \left(\frac{\theta''}{\theta'}\right)^{n/2} e^{-\frac{(\theta''-\theta')}{2\theta'\theta''}\sum^n x_i^2} \leq K$$

OR

$$-\frac{\theta''-\theta'}{2\theta'\theta''}\sum^n x_i^2 \leq K$$

BUT, WE DO NOT KNOW THE POLARITY OF $\theta''-\theta'$, SO THIS IS AS FAR AS WE CAN GO. $\therefore \nexists$ NO UMP TEST.

(9-22) $X_i \sim n(\theta, 100)$ $n = 25$

$\alpha = 0.1$

$H_0: \theta = 75$ vs $H_1: \theta > 75$

UMP TEST IS $\bar{X} > C$

$\bar{X} \sim n(\theta, \frac{100}{25}) = n(\theta, 4)$

UNDER H_0:

$\bar{X} \sim n(75, 4)$

$$\alpha = 0.1 = P_r[H_1 | H_0]$$

$$= \int_c^{\infty} n(75, 4)\,dx$$

$$= \int_{\frac{c-75}{2}}^{\infty} n(0,1)\,dx$$

$$\Rightarrow \int_{-\infty}^{\frac{c-75}{2}} n(0,1) = 0.9$$

$$\therefore \frac{c-75}{2} = 1.282$$

$$c - 75 = 2.564$$

$$\Rightarrow c = 77.564$$

$\therefore$ REJECT H_0 IF $\bar{X} \geq 77.564$

(9.28) $X_i \sim f(x;\theta)$

$f(x;\theta) = e^{p(\theta)K(x) + S(x) + q(\theta)}; \ a<x<b$

$\Omega = \{\theta; \theta > \theta'\}$

(a) $p(\theta)$ INCREASING FUNC OF θ

$H_0: \theta = \theta'$, $H_1: \theta > \theta'$ $= \theta''$

$$\frac{L_0}{L_1} = \frac{e^{p(\theta')\Sigma K(x)} e^{\Sigma S(x)} e^{nq(\theta')}}{e^{p(\theta'')\Sigma K(x)} e^{\Sigma S(x)} e^{nq(\theta'')}} \le k$$

① $[p(\theta') - p(\theta'')] \Sigma K(x) \le k'$

SINCE $p(\theta)$ IS STRICTLY INCREASING;

AND $\theta'' > \theta' \Rightarrow p(\theta'') > p(\theta')$

$\Rightarrow p(\theta') - p(\theta'') < 0$

$\Rightarrow \Sigma K(x) \ge k'' = C$ IS UMP

(b) FROM ①, IF $p(\theta)$ IS DECREASING,

THEN $p(\theta') - p(\theta'') > 0$ AND

$\Sigma K(x) \le k'''$ IS UMP

23. 19
24.
25.
26.

1. 10.12
2. 10.14
3. 10.15
4. 10.38
5. 10.42

6-10 p. 2, HANDOUT

6 → MARKS

11. p.12 handout
12. p.14 "
13. 14 "
14 14 "
15. 17 "
16. Prove Thm 5 on p. 21
17. " " 6 " " "

LET $W_N = \sum_{i=1}^{N} \left(\frac{i}{N}\right) Z_i$ (wilcoxon)

$$S_N = \sum_{i=1}^{N} \left(\frac{i}{N}\right)^2 Z_i$$

$$M_N = \sum_{i=1}^{N} \left(\frac{i}{N} - \frac{N+1}{2N}\right)^2 Z_i$$

$$A_N = \sum_{i=1}^{N} \left|\frac{i}{N} - \frac{N+1}{2N}\right| Z_i \qquad F_Y(x) = F_X(x-\theta)$$

18. Find ARE (W_N, S_N) under location alternatives. Give examples when $F_X(x) = \Phi(x)$ and $F_X(x) = 1 - e^{-x}; x \geq 0$
19. Same, but "scale" for "location"
20. Find ARE (W_N, M_N) under scale alternatives when $F_X(x) = \Phi(x)$
21. Find ARE (A_N, M_N) under scale alt. Give examples when $F_X(x) = \Phi(x)$, $\Phi(x)$ = double exponential, $F_X(x)$ = cauchy
22. Find ARE(M_N, F) under scale alts... Give an example when the ~~the three~~ d.f.'s are as in #21.

(10-1) $n = 10$

$$t_9 = \frac{\sqrt{n}\,\bar{x}}{\left[\sum (x_i - \bar{x})^2/(n-1)\right]^{1/2}} = \frac{\sqrt{10}\;0.6}{\sqrt{3.6}/3}$$

$$= \frac{3\sqrt{10}\;0.6}{\sqrt{3.6}} = 3$$

$\alpha = 0.05 = P_r[H_1 / H_0]$

UNTER H_0, $t_9 \sim T_9$

$$\alpha = 0.05 = \int_{c}^{c} T_9\, dx \Rightarrow \int_{-\infty}^{c} T_9\, dx = 0.975$$

$\therefore\ c = 2.262$

REJECT H_0 SINCE $|3| \geq 2.262$

10-1, 6, 10, 12, 13, 14, 15

4/7 —

(10-6) $X_1, \dots, X_n \sim n(\theta, 1)$

$\Rightarrow H_0: \theta = \theta'$

$H_1: \theta \neq \theta' \xrightarrow[\text{GET}]{\text{WANNA}} |\bar{x} - \theta'| \geq c$

JOINT pdf IS

$$L(\Omega) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\sum^n (x_i - \theta)^2/2}$$

$$\ln L(\Omega) = n \ln \sqrt{\frac{1}{2\pi}} - \sum^n (x_i - \theta)^2/2$$

$$\frac{\partial \ln L(\Omega)}{d\theta} = \sum^n (x_i - \theta) = 0$$

$$\Rightarrow \theta = \frac{1}{n} \sum^n x_i = \bar{x}$$

$$\frac{L(\hat{\omega})}{L(\hat{\Omega})} = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp -\frac{1}{2} \sum^n (x_i - \theta')^2}{\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp -\frac{1}{2} \sum^n (x_i - \bar{x})^2}$$

$$= \exp(-n(\bar{x} - \theta')^2/2$$

$$\lambda \leq \lambda_0$$

$$-\frac{n}{2}(\bar{x} - \theta')^2 \leq \ln \lambda_0$$

$$(\bar{x} - \theta')^2 \geq \frac{2}{n} \ln \frac{1}{\lambda_0} = c'$$

OR

$$|\bar{x} - \theta'| \geq c$$

<u>NOT</u> UMP

(10-10)

$X_1 \dots X_n \sim n(\theta_1, \theta_3)$

$Y_1 \dots Y_m \sim n(\theta_2, \theta_3$

(a) $H_0 : \theta_1 = \theta_2 \; ; \; \theta_3 = \theta_4$

$H_1 :$

$\omega = \{ \theta_1 = \theta_2 \,,\, \theta_3 = \theta_4 \}$

$\Omega = \{ -\infty < \theta_1 , \theta_2 < \infty \,,\, \theta_3 \ne \theta_4 \geq \theta$

$$L(\omega) = \left(\frac{1}{2\pi\theta_3}\right)^{n/2} \left(\frac{1}{2\pi\theta_3}\right)^{n/2} \times \exp\left[\frac{-\Sigma (x_i - \theta_1)^2 + \Sigma (Y_i - \theta_1)^2}{2\theta_3} \right]$$

$$\ln L(\omega) = \frac{n}{2} \ln \frac{1}{2\pi\theta_3} + \frac{m}{2} \ln \frac{1}{2\pi\theta_3} - \frac{1}{2\theta_3}\left[\sum^{n} (x_i - \theta_1)^2 + \sum^{m} (Y_i - \theta_1)^2 \right]$$

$$\frac{\delta \ln L(\omega)}{\delta \theta_1} = \frac{1}{\theta_3}\left[\sum^{n} (x_1 - \theta_1) + \sum^{m} (Y_i - \theta_1) \right] = 0$$

$$U = \frac{n\bar{X} + m\bar{Y}}{n+m}$$

$$\frac{\delta \ln L(\omega)}{\delta \theta_3} = \frac{-(n+m)}{2\theta_3} + \frac{1}{2\theta_3^2}\left[\sum^{n} (x_i - \theta_1)^2 + \sum^{m} (Y_i - \theta_1)^2 \right]$$

$$\hat{\theta}_3 = w = \frac{1}{n+m}\left[\sum^{n} (x_i - U)^2 + \sum^{m} (Y_i - U)^2 \right]$$

GIVES

$$L(\hat{\omega}) = \left(\frac{e^{-1}}{2\pi w} \right)^{\frac{n+m}{2}}$$

NOW

$$\ln L(\Omega) = \frac{n}{2} \ln 2\pi\theta_3 - \frac{m}{2} \ln 2\pi\theta_4 - \sum^{n} (x_i - \theta_1)^2 / 2\theta_3 - \sum^{m} (Y_i - \theta_2)^2 / 2\theta_4$$

MO!

$$\frac{\delta \ln L(\Omega)}{\delta \theta_1} = 0 = \frac{\sum^n (x_i - \theta_1)}{\theta_3} = 0$$

$$\Rightarrow u_1 = \hat{\theta}_1 = \bar{x}$$

$$\frac{\delta \ln L(\Omega)}{\delta \theta_2} = 0 = \frac{\sum^m (Y_i - \theta_2)}{\theta_4} = 0 \Rightarrow$$

$$\Rightarrow U_2 = \hat{\theta}_2 = Y$$

SIMILARLY

$$\omega_1 = \hat{\theta_3} = \frac{1}{n} \sum^n (x_i - \bar{x})^2$$

AND

$$\frac{\delta \ln L(\Omega)}{\delta \theta_4} = \frac{-n}{2\theta_4} + \frac{\sum (Y_i - \theta_2)^2}{2\theta_4{}^2} = 0$$

$$\Rightarrow \omega_2 = \hat{\theta_4} = \frac{1}{m} (Y_i - \bar{Y})^2$$

GIVES

$$\frac{L(\hat{\omega})}{L(\hat{\Omega})} = \frac{\left[\sum^n (x_i - \bar{x})^2/n\right]^{n/2} \left[\sum^m (Y_i - \bar{Y})^2/m\right]^{\frac{m}{2}}}{\left\{\left[\sum^n (x_i - U)^2 + \sum^m (Y_i - U)^2\right] / (m+n)\right\}^{\frac{m+n}{2}}}$$

$$\ni \mu = \frac{n\bar{x} - m\bar{Y}}{n+m}$$

(b) $H_0 : \theta_3 = \theta_4 \ni \theta_1 \;\; \theta_2$ UNSPECIFIED

$L(\hat{\Omega})$ IS SAME AS IN (a)

$$\omega = \{\theta_3 = \theta_4 ; \theta_1, \theta_2 \geq 0\}$$

$$L(\omega) = \left(\frac{1}{2\pi\theta_3}\right)^{\frac{n+m}{2}} e^{-\frac{\sum^n (x_i - \theta_1)^2 + \sum^m (Y_j - \theta_2)^2}{2\theta_3}}$$

$$\ln L(\omega) = \frac{n+m}{2} \ln(2\pi\theta_3)$$
$$- \frac{1}{2\theta_3}\left[\sum^n (x_i - \theta_1)^2 + \sum^m (Y_i - \theta_2)^2\right]$$

$$\frac{\delta \ln L(\omega)}{\delta \theta_1} = \frac{1}{\theta_3} \sum^n (x_i - \theta_1) \Rightarrow u_1 = \hat{\theta}_1 = \bar{x}$$

SIMILARLY

$$\hat{\theta_2} = U_2 = \bar{Y}$$

$$\frac{\delta \ln L(\omega)}{\delta \Theta_3} = \frac{-(n+m)}{2\Theta_3} + \frac{1}{2\Theta_3^2}\Big[\sum^n (x_i - \Theta_1)^2 + \sum^m (Y_i - \Theta_2)^2\Big] = 0$$

ETC.

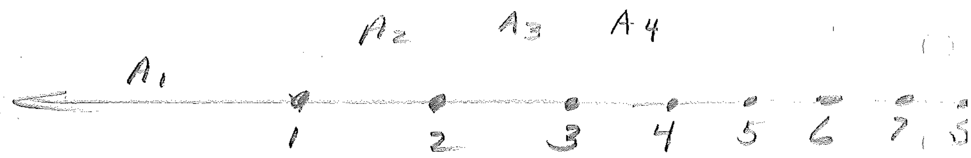
GIVES

$$\frac{L(\hat{\omega})}{L(\hat{\Omega})} = \frac{(n+m)^{\frac{n+m}{2}} \left[\sum^n (X_i - \bar{X})^2\right]^{\frac{n}{2}} \left[\sum^m (Y_i - \bar{Y})^2\right]^{m/2}}{n^{n/2} m^{m/2} \left[\sum^n (X_i - \bar{X})^2 + \sum^m (Y_i - \bar{Y})^2\right]^{\frac{n+m}{2}}}$$

$$= \frac{(m+n)^{\frac{n+m}{2}}}{n^{\frac{n}{2}} m^{m/2}} \left[\frac{1}{\left(1 + \frac{\sum (Y_i - \bar{Y})^2/m-1}{\sum (X_i - \bar{X})^2/n-1}\right)^{n/2}}\right] \xleftarrow{} F$$

$$\times \left[\frac{1}{\left(1 + \frac{\sum^n (X_i - \bar{X})^2/n-1}{\sum^m (Y_i - \bar{Y})^2/m-1} \frac{n-1}{m-1}\right)^{\frac{m}{2}}}\right] \xleftarrow{} F$$

(10-12)



H_0: $n(3,4)$

$$p_{i0} = \int_{A_i} \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-3)^2}{2\cdot 4}}\, dx$$

$$p_1 = P_r(X<0) = P_r\left[\frac{X-3}{2} < -\frac{3}{2}\right] = 0.067$$

$$p_2 = P_r[0<X<1] = P_r[Z<-1] = \overset{0.092}{\cancel{0.162}}$$

$p_3 = 0.150$

$p_4 = 0.191$

$p_5 = 0.191$

$p_6 = 0.150$

$p_7 = 0.092$

$p_8 = 0.067$

CONSIDER $\displaystyle\sum_{i=1}^{8} \frac{(X_i - np_i)^2}{np_i} \sim \chi^2_7 = 6.92$

FAIL TO REJECT IT.

(10-13)

$P[R \,\&\, Y] = 9/16 \qquad P[R \,\&\, G] = 3/16$

$P[W \,\&\, Y] = 3/16 \qquad P[W \,\&\, G] = 1/16$

160 OBSERVATIONS

$R \,\&\, Y = 86 \qquad R \,\&\, G = 26$

$W \,\&\, Y = 35 \qquad W \,\&\, G = 13$

MULTINOMIAL DIST: $n = 4$

$$Q_3 = \sum_{i=1}^{k} \frac{(X_i - np_i)^2}{np_i} \sim \chi^2(k-1)$$

$$\hat{Q}_3 = \frac{\left(\frac{86}{160} - 4\cdot 9/16\right)^2}{4\cdot 9/16} + \frac{\left(\frac{35}{160} - 4\cdot 3/16\right)^2}{4\times 3/16} + \frac{\left(\frac{26}{160} - 4\cdot 3/16\right)^2}{4\times 3/16} + \frac{\left(\frac{13}{160} - 4\cdot \frac{1}{16}\right)^2}{4\times 1/16}$$

$= 1.3034 + 0.3763 + 0.4408 + 0.1139$

$= 2.234$

$Q_3 \sim \chi^2_3$

@ 0.01 $\Rightarrow$ 11.3

$2.23 < 11.3 \Rightarrow$ ACCEPT MENDELIAN HYPOTHESIS.

(10.14)

	A	B	C	D	F	
I	15	25	32	17	11	$\leftarrow p_{i1}$
II	9	18	29	28	16	$\leftarrow p_{i2}$

$H_0: p_{i1} = p_{i2}$

$k = 5$

USE RESULTS OF EXAMPLE II, p. 312:

$$q_{k-1} = \sum_{j=1}^{2} \sum_{i=1}^{k} \frac{[X_{ij} - n_j\{(X_{i1}+X_{i2})/(n_1+n_2)\}]^2}{n_j[(X_{i1}+X_{i2})/n_1+n_2]} \sim \chi^2_{k-1}$$

$n_j = 100, \ j = 1, 2$

CRANKING GIVES:

$q_4 = 6.4$

χ^2_4 @ $\alpha = 0.05 \Rightarrow 9.49$

$6.4 < 9.49 \Rightarrow$ ACCEPT H_0

(10-15)

INCOME	# KIDS 0	1	2	> 2	$X_{i\cdot}$	$P_{i\cdot}$
< 6000	11	24	49	44	128	.45
6-12,000	9	28	39	31	107	.37
> 12,000	6	15	16	15	52	.18
$X_{\cdot j}$	26	67	104	90		

TEST @ 0.1 LEVEL THE HYPOTHESIS THAT INCOME & FAMILY SIZE ARE IND.

$$Q_6 = \sum_{j=1}^{4} \sum_{i=1}^{3} \frac{\left[X_{ij} - \frac{X_{i\cdot}}{N} X_{\cdot j}\right]^2}{\left(\frac{X_{i\cdot}}{N}\right) X_{\cdot j}} \sim \chi^2_{(3-1)(4-1)} = \chi^2_{(6)}$$

$\alpha = 0.1 = Pr[Q_6 \geq c / H_0 \text{ TRUE}]$

$\Rightarrow c = 16.8$

REJECT IF $q_6 > c = 16.8$

TURNS OUT

$q_6 = 3.8 < 16.8 \Rightarrow$ SO ACCEPT H_0

(10-38)

$$x_i \sim n(\beta c_i, \gamma^2 c_i^2);\ i = 1, 2, \ldots, n$$

$$L = \left(\frac{1}{2\pi\gamma^2}\right)^{n/2} e^{-\frac{1}{2}\sum^{n} \frac{(x_i - \beta c_i)^2}{\gamma^2 c_i^2}} \prod^{n}\left(\frac{1}{c_i}\right)^{\frac{n}{2}}$$

$$\ln L = -\frac{1}{2}\sum^{n} \frac{(x_i - \beta c_i)^2}{\gamma^2 c_i^2} - \frac{n}{2}\ln 2\pi\gamma^2 - \frac{n}{2}\ln\left(\prod \frac{1}{c_i}\right)$$

$$\frac{\delta \ln L}{\delta \beta} = 0 = \frac{1}{\gamma^2}\sum^{n} \frac{(x_i - \beta c_i)}{c_i}$$

OR

$$\sum \frac{x_i}{c_i} - n\beta = 0 \Rightarrow \hat{\beta} = \sum \frac{x_i}{n c_i}$$

$$\frac{\delta \ln L}{\delta \gamma} = \frac{1}{\gamma^3}\sum \frac{(x_i - \hat{\beta} c_i)^2}{c_i^2} - \frac{n}{\gamma} = 0$$

$$\Rightarrow \hat{\gamma}^2 = \sum^{n} \frac{(x_i - \hat{\beta} c_i)^2}{n c_i^2}$$

(10-39) $L(\alpha, \beta, \sigma^2; x_1 \ldots x_n) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{\frac{-1}{2\sigma^2}\left[\Sigma(x_i - \alpha) - \beta(c_i - \bar{c})\right]^2}$

$\omega = \{(\alpha, \beta, \sigma^2);\ -\infty < \alpha < \infty,\ \sigma^2 > 0,\ \beta = \beta_0\}$

FOR $L(\Omega) \Rightarrow \hat{\alpha} = \bar{x}$, $\hat{\beta} = \frac{\Sigma(c_i - \bar{c})x_i}{\Sigma(c_i - \bar{c})^2}$

$$\hat{\sigma}^2 = \frac{\Sigma(x_i - \alpha - \beta(c_i - \bar{c}))^2}{n}$$

FOR $L(\omega) \Rightarrow \hat{\alpha}_1 = \bar{x}$,

$$\hat{\sigma}_1^2 = s^2 = \frac{1}{n}\Sigma(x_i - \hat{\alpha})^2$$

THIS GIVES

$$\lambda = L(\hat{\omega}) / L(\hat{\Omega})$$

$$= \frac{\left(\frac{1}{2\pi\hat{\sigma}_1^2}\right)^{n/2} e^{-\frac{1}{2\hat{\sigma}_1^2}(\Sigma(\cancel{x_i - \hat{\alpha}_1}))^2}}{\left(\frac{1}{2\pi\hat{\sigma}^2}\right)^{n/2} e^{-\cancel{\frac{1}{2\hat{\sigma}^2}}\Sigma(\cancel{x_i} - \hat{\alpha} - \hat{\beta}(c_i - \bar{c}))^2}}$$

$$= \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_1^2}\right)^{n/2}$$

REJ H_0 IF $\lambda \le \lambda_0$ $\quad H_0: \beta = 0$

HINT IS $\sum^n (x_i - \hat{\alpha})^2 = n\hat{\sigma}^2 + \hat{\beta}^2 \Sigma(c_i - \bar{c})^2$

FROM (10;30): $\hat{\alpha} \sim n(\alpha, \sigma^2/n) \Rightarrow \frac{\hat{\alpha} - \alpha}{\sigma/\sqrt{n}} \sim n(0,1)$

$\hat{\beta} \sim n\left(\beta, \frac{\sigma^2}{\Sigma(c_i - \bar{c})^2}\right) \Rightarrow \frac{\hat{\beta} - \beta}{\sigma/\sqrt{\sum^n (c - \bar{c})^2}} \sim n(0,1)$

$\frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-2}$

$$\lambda^{2/n} = \hat{\sigma}^2 / \hat{\sigma}_1^2 = \hat{\sigma}^2 \Big/ \frac{\Sigma(x_i - \bar{x})^2}{n}$$

$$= \hat{\sigma}^2 \Big/ \left[n\hat{\sigma}^2 + \hat{\beta}^2 \Sigma(c_i - \bar{c})^2\right]/n$$

$$= \frac{1}{1 + \frac{\hat{\beta}^2 \Sigma(c_i - \bar{c})^2}{n\hat{\sigma}^2}} = \frac{1}{1 + \frac{1}{n}\left(\frac{\hat{\beta}^2}{\hat{\sigma}^2 / \Sigma(c_i - \bar{c})^2}\right)}$$

$$\frac{\frac{\hat{\beta} - \beta}{\sigma/\sqrt{\Sigma(c_i - \bar{c})^2}}}{\sqrt{\frac{n\hat{\sigma}^2}{\sigma^2(n-2)}}} = \frac{\hat{\beta}}{\sqrt{\frac{n}{n-2}}\sqrt{\frac{\hat{\sigma}^2}{\Sigma(c_i - \bar{c})^2}}} \sim t_{n-2}$$

$$\Rightarrow \frac{\hat{\beta}^2}{\frac{n}{n-2}\frac{\hat{\sigma}^2}{\Sigma(c_i - \bar{c})^2}} \sim F_{(1, n-2)}$$

$$\therefore \lambda^{2/n} \sim \frac{1}{1 + F_{(1, n-2)}}$$

(10-42) $\sum_{i=1}^{n} K(x_i, Y_i) \cdots \sum_{n} K_4(x_i Y_i)$

FROM p. 242, Sec. 7-8:

$f(x, Y, \theta_1 \theta_2 \theta_3 \theta_4)$

$$= \exp\left[\sum_{j=1}^{4} p_1(\theta_1\theta_2\theta_3\theta_4) K_j(x_i Y_i) + S(x_i Y_i) + q(\theta_1, \theta_2\theta_3\theta_4)\right]$$

$\rho = 0$

$$\Rightarrow f(x, Y, \theta_1\theta_2\theta_3\theta_4) = \frac{1}{2\pi\theta_3\theta_4}$$

$$\times \exp^{-}\left[\left(\frac{X-\theta_1}{\theta_3}\right)^2 + \left(\frac{Y-\theta_2}{\theta_4}\right)^2\right]$$

$$= \exp\left[-\ln 2\pi\theta_3\theta_4 + \frac{1}{2}\left\{\underbrace{\frac{-X^2}{\theta_3^2}}_{K_3(X)} + \underbrace{\frac{2\theta_1 X}{\theta_3^2}}_{K_1(X)} - \frac{\theta_1^2}{\theta_3^2} - \underbrace{\frac{Y^2}{\theta_4^2}}_{K_4(Y)} \underbrace{+ \frac{2\theta_2 Y}{\theta_4^2}}_{K_2(Y)} - \frac{\theta_2^2}{\theta_4^2}\right\}\right]$$

THIS GIVES

$S(X, Y) = 0$

$$q(\theta_1 \cdots \theta_4) = -\frac{1}{2}\left(\frac{\theta_1^2}{\theta_3^2} + \frac{\theta_2^2}{\theta_4^2} - \ln 2\pi\theta_3\theta_4\right)$$

SUFFICIENT STATS:

$Q_1 = \sum_1^n K_1(x_i) = \sum x_i = n\bar{X}$

$Q_2 = \sum^n Y_i = n\bar{Y}$

$Q_3 = \sum_{i=1}^{n} K_3(Y) = \sum^n Y_i^2$

$Q_4 = \sum^n K_4(Y) = \sum^n Y_i^2$

$\bar{X} = \frac{Q_1}{n}$

$\bar{Y} = Q_2/n$

$V_1 = \sum^n (x_i - \bar{X})^2$

$= Q_3 - Q_1^2/n$

$V_2 = \sum^n (Y_i - \bar{Y})^2$

$= Q_4 - Q_2^2/n$

6. PROB (1) HANDOUT, p. 2

FIND THE pdf OF Y_1 ; OF Y_n

1. $Pr[X_1 > Y_1] = Pr[X_1 > Y_1, X_2 > Y_1, \ldots, X_n > Y]$

$$= [Pr[X > Y_1]]^n$$

$$Pr[X_1 > Y_1] = 1 - G_1(Y_1)$$

$$Pr[X > Y_1] = 1 - F(Y)$$

$$\Rightarrow \quad 1 - G_1(Y_1) = [1 - F(Y)]^n$$

$$G_1(Y_1) = 1 - [1 - F(Y)]^n$$

$$g_1(Y_1) = \frac{d}{dY_1} G_1(Y_1)$$

$$= -n(1 - F(Y))^{n-1}(-f(Y_1))$$

$$= n f(Y_1)[1 - F(Y)]^{n-1}$$

2. $Pr[Y_n \le Y] = Pr[X_1 \le Y, X_2 \le Y, \ldots, X_n \le Y]$

$$G_n(Y) = [F(Y)]^n$$

$$g_n(Y) = n f(Y) F^{n-1}(Y)$$

ALTERNATE WAY:

$g(Y_1, \ldots, Y_n) = n! \prod_{i=1}^{n} f(Y_i) \quad \ni \quad a < Y_1 < Y_2 < \ldots < Y_n < b$

THEN, MARGINAL pdf IS:

$g_n(Y_n) =$

$$\int_{Y_{n-1}=a}^{Y_n} \cdots \int_{Y_2=a}^{Y_3} \int_{Y_1=a}^{Y_2} n! f(Y_1) f(Y_2) \cdots f(Y_n) dY_1 \cdots dY_{n-1}$$

$$\int_a^{Y_n} \cdots \int_a^{Y_4} \int_a^{Y_3} n! F(Y_2) f(Y_2) f(Y_3) \cdots f(Y_n) \, dY_2 \cdots dY_{n-1}$$

INT 1

$$\int_a^{Y_3} F(Y_2) f(Y_2) dY_2 = \frac{1}{2} [F(Y_2)]^2 \Big|_a^{Y_3} = \frac{1}{2} F^2(Y_3)$$

$$g_n(Y_n) = \int_a^{Y_n} \cdots \int_a^{Y_5} \int_a^{Y_4} \frac{n!}{2} F^2(Y_3) f(Y_3) \cdots f(Y_n) dY_3 \cdots dY_{n-1}$$

BUT

$$\int_a^{Y_4} \frac{1}{2} F^2(Y_3) f(Y_3)\, dY_3 = \frac{1}{3 \cdot 2} F^3(Y_3)\Big|_a^{Y_4} = \frac{1}{3!} F^3(Y_4)$$

AND

INT ② $$g_n(Y_n) = \int_a^{Y_n} \cdots \int_a^{Y_6} \int_a^{Y_5} \frac{n!}{6!} F^3(Y_4) f(Y_4) \times f(Y_5) \cdots f(Y_n)\, dY_1 \cdots dY_{n-1}$$

AFTER $n-1$ INTEGRATIONS:

$$g_n(Y_n) = n! \frac{F^{n-1}(Y_n)}{(n-1)!} f(Y_n) = n F^{n-1}(Y_n) f(Y_n)$$

SIMILAR APPROACH FOR $g_1(Y_1)$:

$$g_1(Y_1) = \int_{Y_1}^b \int_{Y_2}^b \cdots \int_{Y_{n-2}}^b \int_{Y_{n-1}}^b n! f(Y_1) f(Y_2) \cdots f(Y_n)\, dY_n\, dY_{n-1} \cdots dY_2$$

$$= \int_{Y_1}^b \int_{Y_2}^b \cdots \int_{Y_{n-3}}^b \int_{Y_{n-2}}^b n! f(Y_1) f(Y_2) \cdots f(Y_{n-1}) [1 - F(Y_{n-1})]\, dY_{n-1} \cdots dY_2$$

$$\int_{Y_{n-2}}^b [1 - F(Y_{n-1})] f(Y_{n-1})\, dY_{n-1} = -\frac{1}{2}[1 - F(Y_{n-1})]^2\Big|_{Y_{n-2}}^b = \frac{1}{2}[1 - F(Y_{n-2})]^2$$

AND

$$g_1(Y_1) = \int_{Y_1}^b \cdots \int_{Y_{n-3}}^b n! f(Y_1) \cdots f(Y_{n-2}) \times \frac{[1 - F(Y_{n-2})]^2}{2}\, dY_{n-2} \cdots dY_2$$

PROCEEDING GIVES

$$g_1(Y_1) = n[1 - F(Y_1)]^{n-1} f(Y_1) \; ; \; a < Y_1 < b$$

7. FIND $g(Y_K)$

$G(Y_K) = P_r[Y_K \le y_K]$

$Y_K \le y_K$ IFF AT LEAST k OF THE RANDOM VARIABLES ARE $\le Y_K$. ASSUME THE i TH TRIAL IS SUCCESS IF $X_i < Y_K$, $i = 1, \ldots, k$

$P_r[\text{SUCCESS}] = \int_a^{Y_K} f(x)\,dx = F(Y_K)$ $\forall$ TRIAL

THEN $G(Y_K) = P_r[Y_K \le Y_K]$

$= P_r$[HAVING AT LEAST k SUCCESSES IN n INDEPENDENT TRIALS].

$$G(Y_K) = \sum_{i=k}^{n} \binom{n}{i} F^i(Y_K)[1-F(Y_K)]^{k-i}$$

$$g(Y_K) = G'(Y_K)$$

$$= \sum_{i=k}^{n} \binom{n}{i} \left\{ i F^{i-1}(Y_K) f(Y_K) [1-F(Y_K)]^{k-i} + F^i(Y_K)(k-i)[1-F(Y_K)]^{k-i-1}(-f(Y_K)) \right\}$$

$$= \binom{n}{k} k F^{k-1}(Y_K) f(Y_K) [1-F(Y_K)]^{n-k} + \frac{n!}{k!(n-k-1)!} F^k(Y_K) [1-F(Y_K)]^{n-k-1}(-f(Y_K))$$

$$+ \frac{n!}{(k)!(n-k-1)!} F^k(Y_K) f(Y_K)[1-F(Y_K)]^{n-k-1}$$

WHEN $i = n$, SECOND TERM $= 0$. THUS, WE ARE LEFT WITH

$$g(Y_K) = \frac{n!}{(k-1)!(n-k)!} F^{k-1}(Y_K)[1-F(Y_K)]^{n-k} f(Y_K)$$

$a < Y < c$.

8. FINDING JOINT DENSITY OF TWO ORDER STATISTICS

$g(Y_1 \ldots Y_N) = N!\, f(Y_1) \cdots f(Y_N) \ni a < Y_i < Y_j < b$

FIND $g_{ij}(Y_i\, Y_j)$

$$g_{ij}(Y_i Y_j) = \int_a^{Y_1} \cdots \int_a^{Y_2} \int_{Y_i}^{Y_j} \int_{Y_{j-2}}^{Y_j} \int_{Y_2}^{b} \cdots \int_{Y_{N-1}}^{b} n!\, f(Y_1) \cdots f(Y_n)\, dY_N \cdots dY_{i-1}$$

NOTE

$$\int_x^Y \left[F(Y) - F(w)\right]^{K-1} f(w)\,dw = \frac{\left[F(x) - F(Y)\right]^K}{K}$$

PLUG & CHUGING & GENERALIZING:

$$g_{ij} = \frac{N!}{(i-1)!\,(j-i-1)!\,(N-j)!} F^{i-1}(Y_i) \times \left[F(Y_j) - F(Y_i)\right]^{j-i-1} \times \left[1 - F(Y_j)\right]^{N-2} f(Y_i)\, f(Y_j)$$

(9. FIND MEDIAN DISTRIBUTION

ASSUME $n \sim$ ODD

$X_1, X_2, \ldots, X_k \sim f(x)$

$$g_k(Y_K) = \frac{n!}{(K-1)!(n-K)!} F(Y_K)^{K-1} \left[1 - F(Y_K)\right]^{n-k}$$

$a < Y_K < b$

LET $k = \frac{n+1}{2}$

FOR n EVEN

LET $Z = \frac{1}{2}\left[X_{\frac{n}{2}} + X_{(\frac{n}{2}+1)}\right]$

$g_i(Y_i, Y_{i+1})$

$Z = \frac{1}{2}(Y_i + Y_{i+1}) \qquad V = Y$

DO A TRANSFORMATIION!

(10) RANGE: NO XFORMATION TO

FIND DISTRIBUTION OF $Y_n - Y_1$

Pg. 14

13. $W_N = \sum^{N} \frac{i}{N} Z_i$ $\quad H_0 : F_X(x) = F_Y(x)$

$m = 3 \quad n = 4$ $\quad \frac{1+2+3}{7}$

X X X Y Y Y Y

X X Y X Y Y Y

35 CASES

1	1	1	1	1	1
2	2	2	2	2	3
3	4	5	6	7	4

$W_N = \frac{6}{7} \quad \frac{7}{7} \quad \frac{8}{7} \quad \frac{9}{7} \quad \frac{10}{7} \quad \frac{8}{7}$

RESULT IS

$p(W_N) = \begin{cases} \frac{1}{35} & W_N = \frac{6}{7} \text{ or } \frac{18}{7} \\ \frac{1}{35} & W_N = 7/7 \text{ or } \frac{17}{7} \\ \frac{2}{35} & 8/7 \text{ or } \frac{16}{7} \\ \frac{3}{35} & 9/7 \text{ or } \frac{15}{7} \\ \frac{4}{35} & 10/7 \text{ or } 14/7 \\ \frac{4}{35} & 11/7 \text{ or } 13/7 \\ \frac{5}{35} & 12/7 \end{cases}$

16. PROVE THEM 5 ON p. 21 (SKIPPED)

17. PROVE THEM 6 ON p. 21

The ARE is the limit of the ratio of sample sizes when the limiting power and sequences of alternatives are the same for both tests, T_N, T_N^*

$$1-\phi(z_\alpha - dc) = 1-\phi(z_\alpha - d^*c^*)$$

OR $dc = d^* c^* \Rightarrow \frac{d^*}{d} = \frac{c^*}{c}$

$$\theta_N = \theta_0 + \frac{d}{\sqrt{N}} = \theta_N^* = \theta_0 + d^*/\sqrt{N^*}$$

$$\Rightarrow \frac{d^*}{d} = \left(\frac{N^*}{N}\right)^{\frac{1}{2}}$$

$$ARE(T, T^*) = \frac{N^*}{N} = \left(\frac{d^*}{d}\right)^2$$

$$= \left(\frac{c}{c^*}\right)^2$$

BUT

$$C = \lim_{N\to\infty} \frac{\frac{dE(T_N)}{d\theta}}{\sqrt{N}\,\sigma(T_N)}\Bigg|_{\theta=\theta_0}$$

$$\therefore ARE(T,T^*) = \left(\frac{c}{c^*}\right)^2 = \lim_{N\to\infty}\left\{\frac{\frac{dE(T_N)}{d\theta}\Big|_{\theta_0}\Big/\sqrt{N}\,\sigma(T_N)\Big|_{\theta_0}}{\frac{dE(T_N^*)}{d\theta}\Big|_{\theta_0}\Big/\sqrt{N}\,\sigma(T_N^*)\Big|_{\theta_0}}\right\}^2$$

18. Find ARE (W_N, S_N) under location alternatives. Give examples when $F_X(x) = \Phi(x)$ and $F_X(x) = 1 - e^{-x}$; $x \geq 0$

$$ARE(W_N, S_N) = \lim_{N \to \infty} \left(\frac{\quad}{\quad} \right)^2$$

UNDER LOCATION

$H_0: F_X(x) = F_Y(Y)$

$H_1: F_Y(x) = F_X(x - \theta)$

FROM p. 22

$$\frac{\left.\frac{dE(W_N)}{d\theta}\right|_{\theta=0}}{\left.\sigma_{W_N}\right|_{\theta=0}} = \frac{\frac{nm}{N}\int_{-\infty}^{\infty} f^2(x)\,dx}{\sqrt{\frac{nm}{12N}}}$$

$$S_N = \sum_{i=1}^N \left(\frac{i}{N}\right)^2 z_i \Rightarrow J(H(x)) = H^2(x)$$

$$H^2(x) = [\lambda_N F_X(x) + (1-\lambda_N) F_Y(x)]^2$$
$$= \lambda_N^2 F_X^2(x) + F_X(x-\theta) \{\lambda_N (1-\lambda_N) F_X(x) + (1-\lambda_N)^2 F_X(x-\theta)\}$$

BY CHERNOFF SAVAGE

$$E\left(\frac{S_N}{m}\right) = \int J(H)\, dF_X(x)$$
$$= \int_{-\infty}^{\infty} H^2(x)\, dF_X(x)$$

$$\left.\frac{dE(S_N)}{d\theta}\right|_{\theta=0} = 2m(1-\lambda_N)^2 \int F_X(x) f^2(x)\,dx + m(1-\lambda_N)\int_{-\infty}^{\infty} F_X(x) f_X(x)\,dx$$

$\int(1)$

$\int(2)$

FROM p. 14:

$$\sigma_{S_N}^2 = \frac{nm}{N}\,\frac{4}{45}$$

PLUGGING:

$$(\text{RATIO})^2 = \left[\frac{nm \int_{-\infty}^{\infty} f_X^2(x)\,dx \quad \sqrt{\frac{4}{45}}\sqrt{\frac{nm}{N}}}{\frac{1}{\sqrt{12}}\sqrt{\frac{nm}{N}}\; 2m\left(\frac{n}{N}\right)^2 \int (1)\,dx + \frac{nm}{N}\int (2)\,dx} \right]^2$$

GIVES

$$ARE = \frac{\frac{48}{45}\left[\int_{-\infty}^{\infty} f_X^2(x)\,dx\right]^2}{\left[\int_{-\infty}^{\infty} F_X(x) f_X(x)\,dx\right]^2}$$

(1) FOR $F(x) = 1 - e^{-x}$; $x > 0$

$\therefore f(x) = e^{-x}$; $x > 0$

GIVES

$ARE(W_N, S_N) = \frac{48}{45} \Rightarrow$ W_N ARE MORE EFFICIENT

(2) FOR $F(x) = \phi(x)$ (?)

$$\frac{48}{45} \int \frac{1}{2\pi} e^{-x^2} dx$$

19(a) FIND $ARE(W_N, S_N)$ UNDER SCALE ALTERNATIVES

$H_0: F_Y(x) = F_X(x)$

$H_1: F_Y(x) = F_X(\theta x)$

FROM THEM 6:

$$ARE(W_N, S_N) = \lim_{N\to\infty}\left[\frac{\frac{dE(W_N)}{d\theta}\Big|_{\theta=\theta_0} \Big/ \sigma_{W_N}\Big|_{\theta=\theta_0}}{\frac{dE(S_N)}{d\theta}\Big|_{\theta=\theta_0} \Big/ \sigma_{S_N}\Big|_{\theta=\theta_0}}\right]^2$$

HERE: $\theta_0 = 1$

NOW

$$W_N = \sum_{i=1}^{N} \frac{i}{N} Z_i$$

FROM CHERNOFF-SAVAGE THEOREM:

$$E\left[\frac{W_N}{m}\right] = \int J[H(x)]\, dF_X(x)$$

$$\Rightarrow J\left(\frac{i}{N}\right) = a_i = \frac{i}{N} \Rightarrow J(x) = x$$

$$\Rightarrow E[W_N] = m\int H(x)\, dF_X(x)$$

$$= m\int [\lambda_N F_X(x) + (1-\lambda_N) F_Y(x)]\, dF_X(x)$$

$$= m\int [\lambda_N F_X(x) + (1-\lambda_N) F_X(\theta x)]\, dF_X(x)$$

$$\frac{dE[W_N]}{d\theta} = m(1-\lambda_N)\,\theta \int f_X(\theta x)\, dF_X(x)$$

$$① \quad \frac{dE[W_N]}{d\theta}\Big|_{\theta_0=1} = \frac{nm}{N}\int f_X(x)\, dF_X(x) = \frac{nm}{N}\int f_X^2(x)\, dx$$

ALSO (p. 13)

$$② \quad \sigma^2_{W_N}\Big|_{\theta=1} = \frac{nm}{12N}$$

SQUARED RANK:

$$S_N = \sum_{i=1}^{N} \left(\frac{i}{N}\right)^2 Z_i \quad ; \quad a_i = \left(\frac{i}{n}\right)^2$$

FROM CHERNOFF-SAVAGE THEOREM:

$$E\left[\frac{S_N}{m}\right] = \int J[H(x)]\, dF_X(x)$$

$$J\left[\frac{i}{N}\right] = \left(\frac{i}{N}\right)^2 \Rightarrow J(x) = x^2$$

$$\Rightarrow E[S_N] = m\int [H(x)]^2\, dF_X(x)$$

$$E(S_N) = m \int [\lambda_N F_X(x) + (1-\lambda_N) F_Y(x)]^2 \, dF_X(x)$$

$$= m \int [\lambda_N F_X(x) + (1-\lambda_N) F_X(\theta x)]^2 \, dF_X(x)$$

$$\frac{dE(S_N)}{d\theta} = 2m \int [\lambda_N F_X(x) + (1-\lambda_N) F_X(\theta x)] \times \theta(1-\lambda_N) f_X(\theta x) \, dF_X(x)$$

WRONG DIFFERENTIATION

$$\left.\frac{dE(S_N)}{d\theta}\right|_{\theta=1} = 2m(1-\lambda_N) \int F_X(x) f_X(x) \, dF_X(x)$$

$$= \frac{2mn}{N} \int_{-\infty}^{\infty} F_X(x) f_X^2(x) \, dx \qquad (3)$$

AND, FROM p. 14:

$$\left.\sigma^2_{S_N}\right|_{\theta=\theta_0} = \frac{nm}{N} \frac{4}{45} \qquad (4)$$

(1) (2) (3) & (4) INTO THEM. 6:

$$ARE(W_N, S_N) = \lim_{N\to\infty} \left[\frac{\frac{mn}{N} \int f_X^2(x)\,dx \Big/ \sqrt{\frac{nm}{12N}}}{\frac{2nm}{N} \int F_X(x) f_X^2(x)\,dx \Big/ \sqrt{\frac{nm}{N}\frac{4}{45}}} \right]^2$$

$$= \lim_{N\to\infty} \left[\frac{\sqrt{12}}{\sqrt{45}} \frac{\int f_X^2(x)\,dx}{\int F_X(x) f_X^2(x)\,dx} \right]^2$$

$$= \lim_{N\to\infty} \left[\sqrt{\frac{4}{9}} \frac{\int f_X^2(x)\,dx}{\int F_X(x) f_X^2(x)\,dx} \right]^2$$

$$= \left[\frac{2}{3} \int f_X^2(x)\,dx \Big/ \int F_X(x) f_X^2(x)\,dx \right]^2$$

CASE 1: $F_X(x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\xi^2/2}$

DEFINE $\phi(x) = \Phi(x) - \frac{1}{2}$

ODD EVEN

$\Rightarrow \Phi(x) = \phi(x) + \frac{1}{2}$

$\phi(x)$ IS ODD FUNCTION.

$$\int F_X(x) f_X^2(x)\,dx$$
$$= \int \left[\phi(x) + \tfrac{1}{2}\right] \frac{1}{2\pi} e^{-x^2} dx$$
$$= \frac{1}{2\pi} \int \cancel{\phi(x) e^{-x^2}} dx + \frac{1}{4\pi} \int e^{-x^2} dx$$
$$= 0 + \frac{\sqrt{2\pi}\,\sqrt{\frac{1}{2}}}{4\pi} \left(\frac{1}{\sqrt{2\pi}\,\frac{1}{\sqrt{2}}} \int e^{-\frac{x^2}{2\cdot\frac{1}{2}}} dx \right)$$
$$= \frac{1}{4\sqrt{\pi}}$$

$$\int f_X^2\,dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2} dx$$
$$= \frac{\sqrt{\pi}}{2\pi} \left[\frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{2}}} \int e^{-\frac{x^2}{2\cdot\frac{1}{2}}} dx \right]$$
$$= \frac{1}{2\sqrt{\pi}}$$

THEN

$$ARE(W_N, S_N) = \left[\frac{2}{3} \frac{\frac{1}{2\sqrt{\pi}}}{\frac{1}{4\sqrt{\pi}}} \right]^2$$
$$= \left[\frac{4}{3}\right]^2 = \frac{16}{9}$$

CASE 2: $F_X(x) = 1 - e^{-x}$

$$\int F_X(x)\, f_X^2(x)\,dx$$

$$= \int (1 - e^{-x})\, e^{-2x}\,dx$$

$$= \int_0^\infty e^{-2x}\,dx - \int_0^\infty e^{-3x}\,dx$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\int f_X^2\,dx = \int_0^\infty e^{-2x}\,dx = \frac{1}{2}$$

THUS:

$$ARE[W_N, S_N] = \left[\cancel{\tfrac{2}{3}} \quad \frac{\frac{1}{2}}{\frac{1}{6}}\right]^2$$

$$= 4$$

21. Find ARE(W_N, M_N) under ~~scale alternatives~~

$$ARE(A_N, M_N) = \left(\text{———} \right)^2$$

$H_0: F_Y(x) = F_X(x)$

$H_1: F_Y(x) = F_X(\Theta X)$

$$T^*_{n,m} = \frac{\sum^n (x_i - \bar{x})}{m-1} \Big/ \frac{\sum^n (Y_i - \bar{Y})^2}{n-1}$$

$$\sim F_{m-1, n-1}$$

$\text{var } X = \Theta^2 \text{var}(Y)$

$$E[T^*_{nm}] = E\left[\frac{\sum (x_i - \bar{x})^2}{m-1}\right]$$

$$E\left[\frac{(n-1)}{\sum (Y_i - \bar{Y})^2}\right]$$

$$= (n-1)\,\text{var } X \; E\left[\frac{1}{\sum (Y - \bar{Y})^2}\right]$$

$$(n-1)\Theta^2 E\left[\frac{\text{var } Y}{\sum (Y_i - \bar{Y})^2}\right]$$

$$= (n-1)\Theta^2 E\left[\frac{1}{V}\right]$$

Turns out

$$E\left[\frac{1}{V}\right] = \frac{1}{n-3}$$

$$E[T^*_{n,m}] = (n-1)\Theta^2/(n-3)$$

SINCE T^*_{nm} IS CENTRAL F: $(\theta = 1)$

$$var(T^*_{mn}) = \frac{2(n-1)^2(m-1+n-1-2)}{(m-1)(n-5)(n-3)2}$$

$$= \frac{2(n-1)^2(N-4)^2}{(m-1)(n-5)(n-3)} \quad ; N = m+n$$

$$\frac{\left[\frac{dE[T^*_{mn}]}{d\theta}\right]^2}{\sigma^2_{T_{nm}}\Big|_{\theta=1}} = \frac{2(m-1)(n-5)}{N-4} \sim \frac{2mn}{N}$$

$$= 2\lambda_N(1-\lambda_N)N$$

$$\lambda_N = \frac{m}{N}$$

FOR MOOD STATISTIC:

$$M_N = \frac{1}{N^2}\sum^N \left(i - \frac{N+1}{2N}\right)^2 z_i$$

$$a_i = \left(\frac{i}{N} - \frac{N+1}{2N}\right)^2$$

$$J[H] = \lim_{n\to\infty} J_n(H_N) = \left(H - \frac{1}{2}\right)^2$$

$$\Longrightarrow J(u) = \left(u - \frac{1}{2}\right)^2$$

$$H(x) = \lambda_n F_x(x) + (1-\lambda_N) F_x(\theta x)$$

BY CHER. SAVAGE:

$$\mu_N = \int_{-\infty}^{\infty} J(H(x))\, f_x(x)\,dx$$

$$= \int [\lambda_N F_x(x) + (1-\lambda_N) F_x(\theta x) - \tfrac{1}{2}]^2 f_x(x)\,dx$$

$$\frac{dE[M_N]}{d\theta}\Big|_{\theta=1} = 2[\lambda_N F_x(x) + (1-\lambda_N) F_x(x) - \tfrac{1}{2}] \times (1-\lambda_N)\, f_x^2(x)\, x\,dx$$

$$= (1-\lambda_N)\int_{-\infty}^{\infty} [2F_x(x) - 1]\, f_x^2(x)\, x\,dx$$

$$= 2(1-\lambda_N)\int \left(F_x(x) - \frac{1}{2}\right) f_x^2(x)\, x\,dx$$

$$\frac{\operatorname{var}(M_N)}{\left(dE(M_N) / \theta_N \right)^2 \Big|_{\theta=\theta_0}} = \lambda_N N (1-\lambda_N) \Big/ 180$$

$$= 720 N \lambda_N (1-\lambda_N) \left\{ \left[\int_{-\infty}^{\infty} [F_X(x) - \tfrac{1}{2}] \times f_X^2(x)\, dx \right]^2 \right\}$$

ASSUME $\Phi(x) = F_X(x)$

$$\int \left(\Phi(x) - \tfrac{1}{2} \right) \phi^2(x)\, x\, dx = \frac{1}{4\pi\sqrt{3}}$$

GIVES

$$ARE(M_N, T_{nm}^*) = \frac{\dfrac{15 N \lambda_N (1-\lambda_N)}{\pi^2}}{2N \lambda_N (1-\lambda_N)}$$

$$= \frac{15}{2\pi^2} < 1$$

-6

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Math 5383 Quiz #1 9/23/76 Name: BOB MARKS

1. (15) Let $f(x) = kx$, $x = 1, 2, 3, 4, 5, 6$, and $f(x) = 0$ elsewhere.

(a) Find k so that $f(x)$ is a p.d.f.

$$\sum_{\text{all } x} f = 1 = K(1+2+3+4+5+6)$$
$$= 21K \Rightarrow K = 1/21$$

(b) If $A = \{x; \frac{1}{2} \le x < 5\}$ find $P(A)$.

$$P[\tfrac{1}{2} \le X < 5] = P[X=1] + P[X=2] + P[X=3] + P[X=4]$$
$$= \frac{1}{21}[1+2+3+4] = \frac{10}{21}$$

2. (10) (a) Determine k so that $f(x) = kx$, $1 \le x \le 6$, and $f(x) = 0$ elsewhere, is a p.d.f.

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_1^6 Kx\,dx = \frac{K}{2}x^2\Big|_1^6 = \frac{K}{2}[36-1]$$
$$= \frac{35K}{2} = 1$$
$$\Rightarrow K = 2/35$$

(b) If $A = \{x; \frac{1}{2} \le x < 5\}$ find $P(A)$.

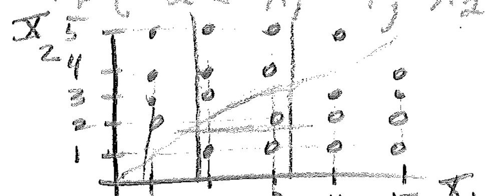
$$P(A) = \int_{\frac{1}{2}}^{1}(0)\,dx + \frac{2}{35}\int_1^5 x\,dx$$
$$= \frac{2}{35}\,\frac{x^2}{2}\Big|_1^5 = \frac{1}{35}[25-1] = \frac{24}{35}$$

3. (10) Consider a box with 5 balls numbered 1, 2, 3, 4, 5. Suppose two balls are drawn at random without replacement. Let X_1 and X_2 denote the numbers on the 1st and 2nd balls, respectively.

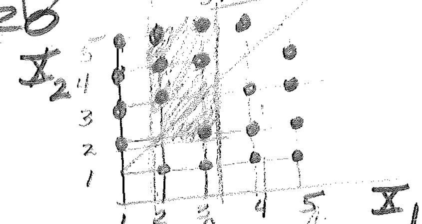
(a) Find a suitable p.d.f. $f(x_1, x_2)$ of X_1 and X_2.

$$f(x_1,x_2) = P(X_1 = n,\ X_2 = m \neq n) = \frac{1}{5}\cdot\frac{1}{4} = \frac{1}{20} \quad ;\ n = 1,2,3,4,5;\ n \neq m = 1,2,3,4,5$$

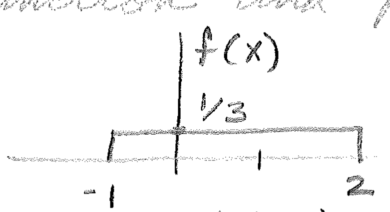
(b) Find $P_r\{2 \le X_1 < 4,\ X_2 > 2\} = 6/20$

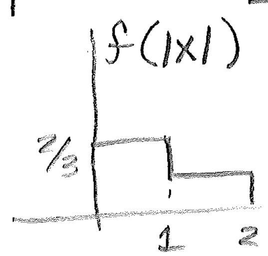


SAMPLE SPACE



4. (15) Let $f(x) = 1/3$, $-1 < x < 2$ and $f(x) = 0$ elsewhere, be the p.d.f. of X. Find the distribution function and p.d.f. of $Y = X^2$.





$$f(|x|) = \begin{cases} 2/3 & ; 0 < x < 1 \\ 1/3 & ; 1 < x < 2 \end{cases}$$

$$F_Y(Y) = P[Y \le Y] = P[X^2 \le Y] = P[|X|^2 \le Y]$$

$$F_Y(Y) = P[|X| \le \sqrt{Y}]$$

$0 \le x < 1$

$$F_Y(Y) = \frac{2}{3}\int_0^{\sqrt{Y}} dx = \frac{2}{3}\sqrt{Y} \quad ; 0 < Y \le 1$$

$F_Y(1) = 2/3$

$1 \le X < 2$

$$F_Y(Y) = \frac{2}{3} + \frac{1}{3}\int_1^{\sqrt{Y}} dx = \frac{2}{3} + \frac{1}{3}[\sqrt{Y} - 1] = \frac{1}{3} + \frac{1}{3}\sqrt{Y} \quad ; 1 < Y \le 4$$

$X > 2$ (i.e. $Y > 4$)

$F_Y(Y) = 0$

DIST. $F_Y(Y)$

$$= \begin{cases} 0 & ; Y < 0 \\ \frac{2}{3}\sqrt{Y} & ; 0 \le Y < 1 \\ \frac{1}{3} + \frac{1}{3}\sqrt{Y} & ; 1 \le Y < 4 \\ 1 & ; Y \ge 4 \end{cases}$$

DIST. SUMMARY

PDF

$$f(Y) = \frac{d}{dY} F_Y(Y) = \begin{cases} \frac{1}{3} Y^{-1/2} & ; 0 < Y < 1 \\ \frac{1}{6} Y^{-1/2} & ; 1 < Y < 4 \\ 0 & ; \text{OTHERWISE} \end{cases}$$

5. (20) Consider $f(x,y) = kx$ if $0<x<1$, $0<y<1$ and $f(x,y) = 0$ elsewhere.

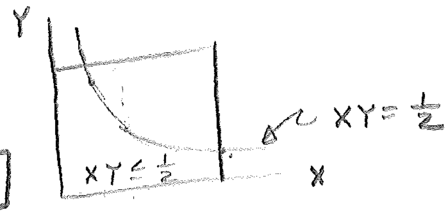
(a) Find k so that $f(x,y)$ is a p.d.f.

$$1 = \int_0^1\int_0^1 kx\,dx\,dY = \int_0^1 \frac{K}{2}x^2\Big|_0^1 dY = \int_0^1 \frac{K}{2}\,dY = \frac{K}{2}$$

$$\Rightarrow K = 2$$

(b) Find $\Pr\{\frac12 < X \le 1,\ 0 \le Y < 1\}$.

$$= \int_0^1\left[\int_{\frac12}^1 2x\,dx\right]dY = \int_0^1 x^2\Big|_{\frac12}^1 dY$$
$$= \int_0^1\left[1-\frac14\right]dY = \int_0^1 \frac34\,dY$$
$$= 3/4$$



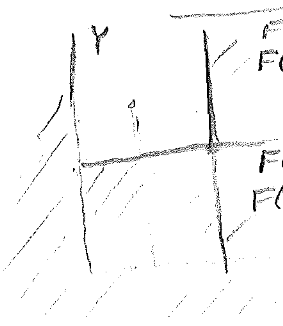
(c) Find $\Pr\{XY \le \frac12\}$. $= 1 - \Pr[XY > \frac12]$

$$1 - \int_{\frac12=Y}^1\int_{X=\frac{1}{2Y}}^1 2x\,dx\,dY = 1 - \int_{1/2}^1 x^2\Big|_{\frac{1}{2Y}}^1 dY$$
$$= 1-\int_{1/2}^1\left[1-\frac{1}{4Y^2}\right]dY = 1-\left[Y+\frac{1}{4Y}\right]_{\frac12}^1$$
$$= 1-\left[\left(1+\frac14\right)-\left(\frac12+\frac12\right)\right] = 1-\frac54+1 = 3/4$$

(d) Find $F(x,y) = 2\int_0^Y\int_0^X x\,dx\,dY = \int_0^Y X^2\,dY = YX^2$; $0<X<1$, $0<Y<1$

OR

$$F(X,Y) = \begin{cases} 0 & ;\ X<0 \text{ \& } Y<0 \\ YX^2 & ;\ 0\le X<1 \text{ \& } 0\le Y<1 \\ 1 & ;\ X\ge 1 \text{ \& } Y\ge 1 \\ X^2 & ;\ Y\ge 1 \text{ \& } 0\le X<1 \\ Y & ;\ X\ge 1 \text{ \& } 0\le Y<1 \end{cases}$$



FOR $Y\ge 1$ $0\le X<1$
$F(Y) = \int_0^1 dY \int_0^X 2x\,dx$
$= X^2$

FOR $X\ge 1$ $0\le Y\le 1$
$F(Y) = \int_0^Y dY \int_0^1 2x\,dx$
$= Y$

6. (20) Find the probability that a poker hand (5 cards) drawn from a deck of 52 cards contains

(a) 2 jacks and 3 queens; $\dfrac{\binom{4}{2}\binom{4}{3}\binom{\cancel{44}}{0}}{\binom{52}{5}}$ 1

(b) 2 jacks; $\dfrac{\binom{4}{2}\binom{\cancel{48}}{\cancel{0}}}{\binom{52}{5}}$ 1 3

-3

(c) at least one club; $= 1 - P[\underline{NO}\ CLUBS]$

$= 1 - \dfrac{\binom{13}{0}\binom{39}{5}}{\binom{52}{5}}$ 1

(d) exactly one pair; divide this by 3!

$\dfrac{\left[\binom{13}{1}\binom{4}{2}\right]\left[\binom{12}{1}\binom{4}{1}\right]\left[\binom{11}{1}\binom{4}{1}\right]\left[\binom{10}{1}\binom{4}{1}\right]}{3!\ \binom{52}{5}}$

-3

TO DISTINGUISH TWINS AA23K AAK23 → 3!

$= \dfrac{13\cdot 12\cdot 11\cdot 10\cdot 4^3\binom{4}{2}}{\binom{52}{5}}$

7. (10) Let B_i, $i \geq 1$ be events.

✓ (a) If $B_1 \subset B_2 \subset B_3 \ldots$ and $B = \bigcup_{i=1}^{\infty} B_i$ prove that

$\lim_{n\to\infty} P(B_n) = P(B)$.

DEFINE

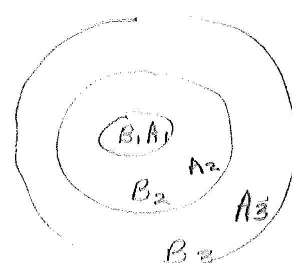
$A_1 = B_1$

$A_2 = B_2 - B_1$

$A_3 = B_3 - B_2$

$\vdots$

$A_K = B_K - B_{K-1}$



i.e. $A_i \cap A_j = \emptyset$, $i \neq j$

NOW, OBVIOUSLY $\bigcup_{K=1}^{n} A_K = \bigcup_{K=1}^{n} B_K = B_n$; ALL A_i DISJOINT

$P[B_n = \bigcup_{K=1}^{n} B_K] = P[\bigcup_{K=1}^{n} A_K] = P[B_1 + (B_2 - B_1) + \ldots + (B_n - B_{n-1})]$

$\therefore \lim_{n\to\infty} P[B_n] = \lim_{n\to\infty} P[\bigcup_{K=1}^{n} A_K] = \lim_{n\to\infty} \sum_{k=1}^{n} P(A_k) = \sum_{k=1}^{\infty} P(A_k) = P(\bigcup_{k=1}^{\infty} A_k)$

BUT $\lim_{n\to\infty} P[\bigcup_{K=1}^{n} A_K] = P[\lim_{n\to\infty} B_n] = P[B]$

$\therefore \lim_{n\to\infty} P[B_n] = P[B]$

Math 53839 97 Quiz #2 A 10/26/76 Name: BOB MARKS

1. Let X have the p.d.f. $f(x) = \theta e^{-\theta x}$, $0 < x < \infty$, $\theta > 0$;

(10) and $f(x) = 0$, elsewhere. (a) Find the moment generating function of X. (b) Find $E(X)$ and $Var(X)$.

(a) $M(t) = E[e^{tX}] = \int_0^\infty \theta e^{-\theta x} e^{tx}\,dx = \theta \int_0^\infty e^{-(\theta - t)x}\,dx$ ✓

$= \dfrac{\theta}{\theta - t}$ ✓ ; $\theta - t > 0$ (OR $\theta > t$)

(b) $\frac{d}{dt} M(t) = \theta \frac{d}{dt}(\theta - t)^{-1} = \theta(\theta - t)^{-2} \Rightarrow \mu = \frac{d}{dt}M(0) = \frac{\theta}{\theta^2} = \frac{1}{\theta}$ ✓

$\frac{d^2}{dt^2} M(t) = \theta \frac{d}{dt}(\theta - t)^{-2} = \theta(-2)(-1)(\theta - t)^{3}$

$= \dfrac{2\theta}{(\theta - t)^3}$

$E(X^2) = \frac{d^2}{dt^2} M(0) = \dfrac{2\theta}{\theta^3} = \dfrac{2}{\theta^2}$

$\Rightarrow \sigma^2 = E(X^2) - E^2(X)$ ✓

$= \dfrac{2}{\theta^2} - \dfrac{1}{\theta^2} = \dfrac{1}{\theta^2}$ ✓

2. If X is a random variable such that $E(X) = 3$ and $E(X^2) = 13$ use Chebyshev's inequality to determine

(10) a lower bound for $Pr(-2 < X < 8)$.

$Pr[|X - \mu| \geq k\sigma] \leq \frac{1}{k^2}$

$Pr[-2 < X < 8] = Pr[-2 - 3 < X - \mu < 8 - 3]$

$= Pr[-5 < X - \mu < 5] = Pr[|X - \mu| < 5]$

NOW $P[|X - \mu| \geq 5] \Rightarrow k\sigma = 5$

$\sigma^2 = 13 - 9 = 4 \Rightarrow \sigma = 2 \Rightarrow k2 = 5 \Rightarrow k = 5/2$

$P[|X - \mu| \geq 5] \leq \frac{4}{25}$

$\Rightarrow -P[|X - \mu| \geq 5] \geq -4/25$

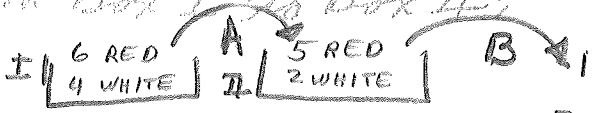
$1 - P[|X - \mu| \geq 5] = P[|X - \mu| < 5] \geq 1 - \frac{4}{25} = \frac{21}{25}$

$Pr[-2 < X < 8] \geq 21/25$ ✓

3. Box I contains 6 red and 4 white balls. Box II contains 5 red and 2 white balls. A ball is selected at random from Box I and placed in Box II after which a ball is chosen at random from Box II.

(a) Find the probability that the ball chosen from Box II is red.

(b) Given that the ball chosen from Box II is red, find the probability that a white ball was transferred from Box I to Box II.



(b) $P[A=w / B=r] = \dfrac{P[A=w]P[B=r/A=w]}{P[A=w]P[B=r/A=w] + P[A=R]P[B=r/A=r]}$

$P[A=w] = 4/10 = \frac{2}{5}$ | $P[B=r/A=w] = 5/8$

$P[A=R] = 6/10 = 3/5$ | $P[B=r|A=r] = \frac{6}{8}$

$\Rightarrow P[A=w|B=r] = \dfrac{2/5 \cdot 5/8}{\frac{2}{5}\cdot\frac{5}{8} + \frac{3}{5}\cdot\frac{6}{8}} = \dfrac{10}{10+18} = \dfrac{10}{28} = \dfrac{5}{14}$

(a) $P[B=r] = P[\{B=r|A=r\} \cup \{B=r|A=w\}]$

$= P[B=r|A=r] + P[B=r|A=w] - \underbrace{P[\{B=r|A=r\} \cap \{B=r|A=w\}]}_{\text{DISJOINT}}$

$= \frac{6}{8} + \frac{5}{8} - \frac{6}{8}\cdot\frac{5}{8}$

$= \frac{6}{8} + \frac{5}{8} - \frac{30}{64} = \dfrac{48+40-30}{64} = \dfrac{58}{64} = \dfrac{29}{32}$

-2

(10) (a) If X and Y are independent show that $\rho = 0$.

$\rho = \dfrac{cov}{\sigma_1 \sigma_2}$

$cov = E[XY] - \mu_1 \mu_2$

IF X & Y ARE INDEPENDENT, THEN $E[XY] = E(X)E(Y) = \mu_1 \mu_2$

$\Rightarrow cov = 0 \Rightarrow \rho = 0$

(b) Is the converse true? NO (Consider $f(x,y) = 1/4$ on (1,1), (2,4), (-1,1), and (-2,4) and zero elsewhere).

$cov(X,Y) = E[XY] - E(X)E(Y)$

$= \frac{1}{4}[1 + \cancel{8} - 1 - 8] - [\frac{1}{4}(\cancel{1+2-1-2})^{0}]\frac{1}{4}[1+4+1-4] = 0$

$\Rightarrow \rho = 0$

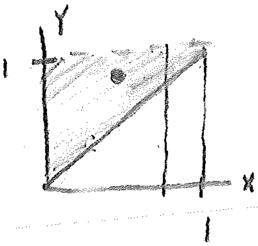
$f_1(x) = \frac{1}{4}$; $x = 1, 2, -1, -2$

$f_2(y) = \frac{1}{2}$; $y = 1, 4$

AND, OBVIOUSLY $f(xy) \neq f_1(x) f_2(y)$

5. Let $f(x,y) = 8xy$, $0<x<y<1$, and $f(x,y)=0$, elsewhere.

(a) Find $f_1(x)$ and $f_2(y)$.



$$f_1(x) = \int_Y f(x,Y)\,dY = \int_x^1 8xY\,dY = 8x\int_x^1 Y\,dY = 4xY^2\Big|_x^1 = 4x(1-x^2)\ ;\ 0<x<1$$

$$f_2(Y) = \int_x f(x,Y) = \int_0^Y 8xY\,dx = 8Y\int_0^Y x\,dx = 4Yx^2\Big|_0^Y = 4Y^3\ ;\ 0<Y<1$$

(b) Find $f(y|x)$ and $f(x|y)$.

$$f(Y|x) = \frac{f(x,Y)}{f_1(x)} = \frac{\cancel{8}2\cancel{x}Y}{\cancel{4x}(1-x^2)} = \frac{2Y}{(1-x^2)}\ ;\ 0<x<Y<1$$

$$f(x|Y) = \frac{f(xY)}{f_2(Y)} = \frac{\cancel{8}2x\cancel{Y}}{\cancel{4}Y^{\cancel{3}2}} = \frac{2x}{Y^2}\ ;\ 0<x<Y<1$$

(c) Find $E(Y|x) = \int_Y Y f(Y|x)\,dY = \dfrac{2\int_x^1 Y^2\,dY}{(1-x^2)} = \dfrac{2Y^3\big|_x^1}{3(1-x^2)}$

$$= \frac{2(1-x^3)}{3(1-x^2)}\ ;\ 0<x<1$$

(d) Find $Var(Y|x)$. $E[Y^2|x] = \dfrac{2\int_x^1 Y^3\,dY}{1-x^2} = \dfrac{Y^4\big|_x^1}{2(1-x^2)}$

$$= \frac{1-x^4}{2(1-x^2)}$$

$$Var(Y|x) = E[Y^2|x] - E^2[Y|x] = \frac{1}{(1-x^2)}\left[\frac{1-x^4}{2} - \frac{4}{9}\frac{(1-x^3)^2}{(1-x^2)}\right]\ ;\ 0<x<1$$

-1

(e) Find ρ.

$E(X) = 8\int_0^1 x^2\int_{y=x}^1 Y\,dY\,dx = 4\int_0^1 x^2Y^2\Big|_x^1 dx = 4\int_0^1 x^2(1-x^2)\,dx = 4\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = 4\left[\frac{1}{3}-\frac{1}{5}\right] = 4\,\frac{2}{15} =$

$= 4\int_0^1 x^2 \cdot x^4 = 4/5$

$E[Y] = 8\int_{Y=0}^1 Y^2\int_{x=0}^Y x\,dx = 4\int_{Y=0}^1 Y^4\,dY$

$4\left[\frac{1}{4}-\frac{1}{6}\right] = 4\cdot\frac{1}{12} = \frac{1}{3}$

$E[X^2] = 4\int_0^1 x^3(1-x^2)\,dx = 4[x^3 - x^5]$

$\Rightarrow \sigma_x^2 = 2/3$

$E[Y^2] = 4\int_0^1 Y^5\,dY = 4/6 = 2/3$

OVER ⟹

(f) Find $Pr\left(\frac{1}{2} \le Y \le 1 \mid X = 3/4\right)$.

$$Pr\left[\tfrac{1}{2} \le Y \le 1 \mid X = \tfrac{3}{4}\right] = \int_{1/2}^1 f\left[Y \mid X=\tfrac{3}{4}\right]dY = \int_{1/2}^1 \frac{2Y\,dY}{\left[1-\left(\frac{3}{4}\right)^2\right]}$$

BUT $Y > X$

$$\Rightarrow Pr\left[\tfrac{1}{2} \le Y \le 1 \mid X = 3/4\right] = \int_{3/4}^1 \frac{2Y}{1-9/16}\,dY = \frac{2}{7/16}\,\frac{1}{2}Y^2\Big|_{3/4}^1$$

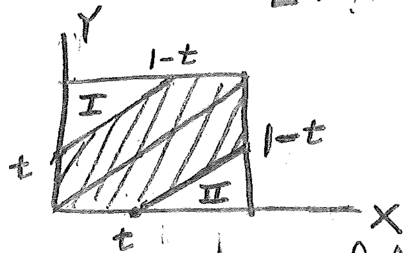
$$= \frac{16}{7}\left[1-\frac{9}{16}\right] = \frac{16}{7}\cdot\frac{7}{16} = 1$$

(WHICH IS RIGHT, SINCE, GIVEN $X = 3/4$, WE ARE ASSURED $Y > X$; THUS, WE'RE CERTAIN THAT $3/4 < Y < 1$.

6. Suppose two numbers, X and Y, are chosen at random and independently of each other in the interval $(0,1)$. Find the probability that they are within t units of each other.

(1)

FIND $P[|X - Y| \leq t] \qquad 0 < t < 1$



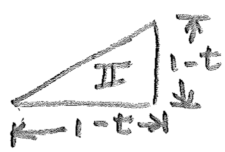
$$f(X) = 1 \quad ; \ 0 < X < 1$$

$$\Rightarrow f(X,Y) = 1 \quad ; \ 0 < X, Y < 1$$

$$P[|X-Y| \leq t] = \iint_{\bigcirc} dX\,dY$$

$$= 1 - \left[\iint_{I} dX\,dY + \iint_{II} dX\,dY\right]$$

$$= 1 - 2\iint_{II} dX\,dY$$



$$\text{AREA (II)} = \tfrac{1}{2}(1-t)^2$$

$$\Rightarrow P[|X-Y| \leq t] = 1 - (1-t)^2 \quad ; 0 \leq t \leq 1$$

7. What is a necessary and sufficient condition for the events A and B to be independent?

(10)

$$P_r[A \cap B] = P_r[A]\,P_r[B]$$

BOB MARKS

100

pg. 94
(3-18)

3 DICE THROWN 10 TIMES

A = OUTCOME OF THROWING SINGLE DIE

$P[A] = \frac{1}{6}$; $A = 1, 2, 3, 4, 5, 6$

THUS, FOR THROWING THREE DICE

$$P[A_1, A_2, A_3] = P[A_1]P[A_2]P[A_3]$$
$$= 1/6^3 \quad ; A_i = 1, 2, 3, 4, 5, 6 \quad , i = 1, 2, 3$$

LET R = EVENT $A_1 = A_2 = A_3$

THEN $$P[R] = \sum_{A_1 = A_2 = A_3} p[A_1, A_2, A_3]$$
$$= \frac{6}{6^3} = \frac{1}{36}$$

LET Q = EVENT ONLY (OR EXACTLY) TWO FACES ALIKE
$= [(A_1 = A_2 \neq A_3)$ OR $(A_1 \neq A_2 = A_3)$ OR $(A_1 = A_3 \neq A_2)]$

$$P(Q) = 3 \times P[A_1 = A_2 \neq A_3]$$
$$= 3 \sum_{A_1 = A_2 \neq A_3} p[A_1, A_2, A_3]$$
$$= 3 \; \frac{36 - 6}{6^3} = \frac{90}{6^3} = \frac{15}{36}$$

X = NUMBER OF TIMES R OCCURS IN 10 THROWS
Y = " " " Q " " 10 "

THIS IS A TRINOMIAL DISTRIBUTION:

$$f(x, y) = \frac{n!}{x!\,y!\,(n - x - y)!} P_R^{x} P_Q^{y} (1 - P_R - P_Q)^{n-x-y}$$
$$= \frac{10!}{x!\,y!\,(10 - x - y)!} \left(\frac{1}{36}\right)^x \left(\frac{15}{36}\right)^y \left(\frac{20}{36}\right)^{10-x-y}$$

$x = $ —
$y = $ —
etc.

BOB MARKS

TO FIND $E[6XY]$, USE MGF:

$$M(t_1,t_2) = (p_1 e^{t_1} + p_2 e^{t_2} + p_3)^n$$
$$= \left(\frac{1}{36} e^{t_1} + \frac{15}{36} e^{t_2} + \frac{20}{36}\right)^{10}$$

$$\frac{\delta M(t_1,t_2)}{\delta t_1} = 10 \left(\frac{1}{36} e^{t_1}\right)\left(\frac{1}{36} e^{t_1} + \frac{15}{36} e^{t_2} + \frac{20}{36}\right)^9$$

$$\frac{\delta^2 M(t_1,t_2)}{\delta t_1 \delta t_2} = 90 \left(\frac{1}{36} e^{t_1}\right)\left(\frac{15}{36} e^{t_2}\right) \times \left(\frac{1}{36} e^{t_1} + \frac{15}{36} e^{t_2} + \frac{20}{36}\right)^8$$

THEN

$$E[XY] = \frac{\delta^2 M(0,0)}{\delta t_1 \delta t_2}$$
$$= 90 \times \frac{1}{36} \times \frac{15}{36}$$

AND

$$E[6XY] = 6 \times 90 \times \frac{1}{36} \times \frac{15}{36}$$
$$= \frac{1}{6} \times \frac{15}{6} \times 15$$
$$= \left(\frac{15}{6}\right)^2$$
$$= \left(\frac{5}{2}\right)^2$$
$$= 25/4$$

BOB MARKS

pg. 98

(3.21) 13.5 % OF PAGES HAD NO TYPING ERROR

LET X = # OF ERRORS ON A PAGE

$X \sim \mu^x e^{-\mu}/x!$

$P[X=0] = \mu^0 e^{-\mu}/0! = 0.135$

FROM TABLE ON pg. 398, IT FOLLOWS THAT

$\mu = -\ln 0.135 = 2.00$

THUS

$P[X=1] = \dfrac{2^1 e^{-2}}{1!} = 0.271$

THIS CAN ALSO BE OBTAINED FROM THE POISSON TABLE ON Pg 398:

$P[X=1] = P[X \le 1] - P[X \le 0]$

$= 0.406 - 0.135 = 0.271$

BOB MARKS

pg. 98

(3.25) X = # CHIPS IN A COOKIE

$f(x)$ = POISSON

WE REQUIRE THAT

$P[X \geq 2] > 0.99$

OR $1 - P[X < 2] > 0.99$

$1 - P[X \leq 1] > 0.99$

$P[X \leq 1] < 0.01$

FROM POISSOIN TABLE ON pg. 398:

$P[X \leq 1] = 0.017$ FOR $\mu = 6$

$P[X \leq 1] = 0.007$ FOR $\mu = 7$

THUS, WE REQUIRE THAT $\mu = 7$ ✓

$\mu \approx 6.63$

BOB MARKS

pg. 103

(3.33) $f(x_i) = e^{-x_i}$, $0 < x_i < \infty$, $i = 1, 2, 3$

X_1, X_2, X_3 ARE INDEPENDENT

$Y = \min(X_1, X_2, X_3)$

NOW

$$P[Y \geq y] = P[X_1 \geq y,\ X_2 \geq y,\ X_3 \geq y]$$
$$= P[X_1 \geq y]\ P[X_2 \geq y]\ P[X_3 \geq y]$$
$$= [P[X \geq y]]^3$$
$$= \left[\int_y^\infty e^{-x}\,dx\right]^3$$
$$= \left[-e^{-x}\Big|_y^\infty\right]^3$$
$$= (e^{-y})^3 = e^{-3y}$$

$$F(y) = P[Y \leq y] = 1 - P[Y \geq y]$$
$$= 1 - e^{-3y} \quad ; 0 < y < \infty$$

$$f(y) = \frac{d}{dy} F(y) = 3e^{-3y} \quad ; 0 < y < \infty$$

BOB MARKS

pg. 110

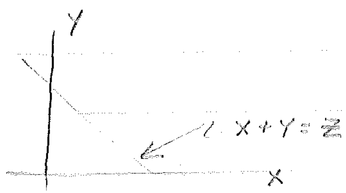
(3.53) $\quad X \sim n(0,1) \qquad Y \sim n(0,1)$

$Z = X + Y$

X AND Y ARE STATISTICALLY INDEPENDENT

$\Rightarrow f(x,y) = f(x)\,f(y)$

$= \frac{1}{2\pi} e^{-(x^2+y^2)/2}$



IN GENERAL, FOR INDEPENDENT X AND Y

$$F(z) = P_r[Z \le z] = \int_{-\infty}^{\infty} f_1(x) \int_{Y=-\infty}^{z-x} f_2(y)\,dy\,dx$$

$$= \int_{X=-\infty}^{\infty} f_1(x)\,F_2(z-x)\,dx$$

$$f(z) = \frac{d}{dz}F(z) = \int_{X=-\infty}^{\infty} f_1(x)\,f_2(z-x)\,dx \Leftarrow \text{CONVOLUTION}$$

NOW $M_Z(t) = E[e^{tZ}]$

$$= \int_{-\infty}^{\infty} f_1(x)\left[\int_{-\infty}^{\infty} f_2(z-x)\,e^{zt}\,dz\right]dx$$

$$= \int_{-\infty}^{\infty} f_1(x) \int_{-\infty}^{\infty} f_2(z) e^{tz}\,dz\; e^{tx}\,dx$$

$$= \left[\int_{-\infty}^{\infty} f_1(x) e^{tx}\,dx\right]\left[\int_{-\infty}^{\infty} f_2(z) e^{tz}\,dz\right]$$

$$= M_1(t)\,M_2(t)$$

NOW, SINCE BOTH X & Y ARE $n(0,1)$:

$$M_1(t) = M_2(t) = e^{\mu t + \sigma^2 t^2/2} = e^{t^2/2}$$

$$\Rightarrow M_Z = \left(e^{t^2/2}\right)^2 = e^{t^2} = e^{(2t^2)/2}$$

THUS $Z \sim n(0,2)$ OR

$$f(z) = \frac{1}{\sqrt{2\pi}\,\sqrt{2}}\, e^{-\frac{z^2}{2(2)}}$$

BOB MARKS

pg. 114

(3.54) $\mu_1 = 3 \quad \mu_2 = 1 \quad \sigma_1^2 = 16 \quad \sigma_2^2 = 25 \quad \rho = 3/5$

(c) FIND $\Pr(-3 < X < 3)$

$X \sim n(3, 16)$

$$\Pr(-3 < X < 3) = \Pr\left[\frac{-3-3}{4} < \frac{X-\mu_1}{\sigma_1} < \frac{3-3}{4}\right]$$
$$= \Pr\left[-1.5 < \frac{X-\mu_1}{\sigma_1} = Z < 0\right]$$
$$= \Pr[0 < Z < 1.5]$$
$$= \Pr[Z < 1.5] - P[Z < 0]$$
$$= \Pr[Z < 1.5] - \tfrac{1}{2}$$

FROM TABLE ON pg. 400:

$$\Pr[Z < 1.5] = 0.933$$

$$\Rightarrow \Pr[-3 < X < 3] = 0.933 - 0.5 = 0.433$$

(d) $X|Y \sim N\left(\mu_1 + \rho \frac{\sigma_1}{\sigma_2}(Y - \mu_2),\ \sigma_1^2(1-\rho^2)\right)$

$$X|Y=-4 \sim N\left(3 + \frac{3}{5}\,\frac{4}{5}(-4-1),\ 16\left(1 - 9/25\right)\right)$$
$$N\left(3 - \frac{12}{5},\ (16)^2/25\right) = N\left(3/5,\ (16)^2/25\right)$$

$$\Pr[-3 < X < 3] = \Pr\left[\frac{-3 - 3/5}{16/5} < \frac{X-\mu}{\sigma} = Z < \frac{3 - 3/5}{16/5}\right]$$
$$= \Pr\left[\frac{-15-3}{16} < Z < \frac{15-3}{16}\right]$$
$$= \Pr\left[\frac{-9}{8} < Z < \frac{3}{4}\right]$$
$$= \Pr[Z < 3/4] - \Pr[Z < -9/8]$$
$$= \Pr[Z < 3/4] - [1 - \Pr[Z < 9/8]]$$
$$= \Pr[Z < 0.75] + \Pr[Z < 1.125] - 1$$
$$= 0.77 + 0.87 - 1 = 0.64$$

(THESE VALUES TAKEN FROM NORMAL TABLE ON p. 400)

BOB MARKS

pg. 114

(3-55)

$f(X,Y) = \frac{1}{2\pi} e^{-\frac{1}{2}(X^2+Y^2)} \left[1 + XY e^{-\frac{1}{2}(X^2+Y^2-2)} \right]$

SHOW $f(X,Y)$ IS A MARGINAL pdf

- IS $f(X,Y) \geq 0$?

OBVIOUSLY, $f(0,0) > 0$

ALSO $\frac{1}{2\pi} e^{-\frac{1}{2}(X^2+Y^2)} \geq 0$

$\therefore f(X,Y)$ CAN BE NEGATIVE ONLY IF

$p(X,Y) = 1 + XY e^{-\frac{1}{2}(X^2+Y^2-2)}$ IS NEGATIVE

NOW $p(0,0) = 1 \geq 0$

CHECK FOR MINIMUMS OF $p(X,Y)$

$\frac{\delta p(X,Y)}{\delta X} = [Y - (XY)(X)] e^{-\frac{1}{2}(X^2+Y^2-2)} = 0$

$\Rightarrow X^2 = 1$

DUE TO SYMMETRY, CHECK FOR MINIMA AT $(X,Y) = (-1,1)$ OR $(1,-1)$

($p(X,Y)$ IS OBVIOUSLY POSITIVE $\forall (X,Y) > (0,0)$)

$p(1,-1) = p(-1,+1) = 0$

THUS, SINCE THE MINIMUM OF $p(X,Y)$ IS ZERO, $p(X,Y) \geq 0 \;\forall\; \{X,Y \mid -\infty < X,Y < \infty\}$

IT FOLLOWS THAT

$f(X,Y) = \frac{1}{2\pi} e^{-\frac{1}{2}(X^2+Y^2)} p(X,Y) \geq 0$

- IS $\iint_{-\infty}^{\infty} f(X,Y)\,dX\,dY = 1$?

$\frac{1}{2\pi} \iint_{-\infty}^{\infty} e^{-\frac{1}{2}(X^2+Y^2)} \left[1 + XY e^{-\frac{1}{2}(X^2+Y^2-2)} \right] dX\,dY$

$= \frac{1}{2\pi} \iint_{-\infty}^{\infty} e^{-\frac{1}{2}(X^2+Y^2)} dX\,dY$

$+ \frac{1}{2\pi} \iint_{-\infty}^{\infty} XY e^{-\frac{1}{2}(2X^2+2Y^2)} dX\,dY\, e^{+1}$

NOW $\frac{1}{2\pi} \iint_{-\infty}^{\infty} e^{-\frac{1}{2}(X^2+Y^2)} dX\,dY = 1$

(THIS IS THE EXPRESSION FOR A BIVARIATE NORMAL pdf WITH $\sigma_1 = \sigma_2 = \mu_1 = \mu_2 = 1$ AND $\rho = 0$, WHICH MUST INTEGRATE TO UNITY)

BOB MARKS

NOW

$$\frac{1}{2\pi}\iint_{-\infty} XY\, e^{-\frac{1}{2}(2X^2+2Y^2)}\, e\, dx\,dY$$

$$= \frac{e}{2\pi}\left[\int_{-\infty}^{\infty} X e^{-X^2} dx\right]^2$$

BUT X IS AN ODD FUNCTION, AND e^{-X^2} IS EVEN $\Rightarrow Xe^{-X^2}$ IS ODD AND $\int_{-\infty}^{\infty} Xe^{-X^2}dx = 0$

THUS

$$\iint_{-\infty}^{\infty} f(x,Y)\,dx = 1 + 0 = 1$$

- SHOW EACH MARGINAL PDF IS NORMAL

$$f_1(x) = \int_{-\infty}^{\infty} f(x,Y)\,dY$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+Y^2)}\left[1 + XYe^{-\frac{1}{2}(X^2+Y^2-2)}\right]dY$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Y^2}dY$$

$$+ \frac{1}{2\pi} X e^{-\frac{1}{2}(2X^2)}\, e \int_{-\infty}^{\infty} Y e^{-\frac{1}{2}(2Y^2)}dY$$

NOW $\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Y^2}dY = 1$

AND, BY THE PREVIOUS "EVEN-ODD" ARGUMENT

$$\int_{-\infty}^{\infty} Y e^{-Y^2}dY = 0$$

THUS $f_1(x) = \frac{1}{\sqrt{2\pi}} e^{-X^2/2}$

BY SYMMETRY,* IF FOLLOWS THAT:

$$f_2(Y) = \frac{1}{\sqrt{2\pi}} e^{-Y^2/2}$$

THUS, BOTH X AND Y ARE $n(0,1)$

* $f(x,Y) = f(Y,x)$

BOB MARKS

8. SUPPOSE THAT 100 CARDS MARKED 1, 2, ..., 100 ARE RANDOMLY ARRANGED IN A LINE. SHOW THAT THE NUMBER OF EVEN INTEGERS IN THE FIRST 20 POSITIONS HAS A HYPERGEOMETRIC DISTRIBUTION WITH PARAMETERS $n = 20, M = 50, N = 100$

THE HYPERGEOMETRIC DISTRIBUTION IN GENERAL MAY BE WRITTEN

$$f(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

HERE, OUR RANDOM VARIABLE IS
X = # OF EVEN NUMBERS ON FIRST 20 CARDS.
TO SHOW X IS HYPERGEOMETRIC, WE SHALL USE THE RELATIVE EFFICIENCY APPROACH. SINCE SAMPLING IS DONE RANDOMLY WITHOUT REPLACEMENT, ALL EVENTS CONTAINED IN OUR SAMPLE SPACE ARE EQUALLY LIKELY TO OCCUR. THE NUMBER OF SUCH EVENTS (WITHOUT REGARD TO ORDER) IS

$$\binom{N}{n} = \binom{100}{20} = \frac{100!}{20!\,80!}$$

FOR A GIVEN $x = X$, WE MUST DIVIDE THIS SAMPLE SPACE INTO DISJOINT REGIONS OF "SUCCESS" AND "FAILURE". A

BOB MARKS

SUCCESS IS THE EVENT THAT THERE ARE EXACTLY X EVEN NUMBERS IN THE CHOSEN TWENTY. A SUCCESS ALSO MUST BE ACCOMPANIED BY X-20 ODD NUMBERS. THERE ARE M = 50 PLACES TO GET EVEN NUMBERS AND N - M = 50 PLACES TO GET ODD NUMBERS. THUS, FOR A GIVEN X, THE NUMBER OF SUCCESSFUL EVENTS IN OUR SAMPLE SPACE IS

$$\binom{M}{X}\binom{N-M}{n-X} = \binom{50}{X}\binom{50}{20-X}$$

THE PROBABILITY THAT A SUCCESS OCCURS (BY RELATIVE FREQUENCY DEFINITION) IS THUS

$$(*)\quad f(x) = \frac{\#\text{ OF SUCCESSES IN SAMPLE SPACE}}{\#\text{ OF ELEMENTS IN SAMPLE SPACE}}$$

$$= \binom{M}{X}\binom{N-M}{n-X} \Big/ \binom{N}{n}$$

$$= \frac{\binom{50}{X}\binom{50}{20-X}}{\binom{100}{20}}$$

$X = \cdot$ —

THIS, OF COURSE, IS HYPERGEOMETRIC

* AGAIN, ON A SINGLE TRIAL, EACH EVENT IN THE SAMPLE SPACE IS ASSUMED EQUALLY LIKELY TO OCCUR.

9. A PARTICLE PERFORMS A "RANDOM WALK" OVER THE POSITIONS $0, \pm 1, \pm 2, \ldots$ IN THE FOLLOWING WAY: THE PARTICLE STARTS AT ZERO. IT MAKES SUCCESSIVE ONE-UNIT STEPS THAT ARE MUTUALLY INDEPENDENT, EACH STEP IS TO THE RIGHT WITH PROBABILITY p, OR TO THE LEFT WITH PROBABILITY $1-p$. LET X BE THE POSITION OF THE PARTICLE AFTER n STEPS

a) SHOW THAT $\frac{X+n}{2}$ HAS A BINOMIAL DISTRIBUTION WITH PARAMETERS n AND p.

b) SHOW THAT THE EXPECTED POSITION OF THE PARTICLE AFTER n STEPS IS $n(2p-1)$.

(a) DEFINE THE RANDOM VARIABLE S WHICH IS THE OUTCOME OF A BINARY TRIAL:

$$P_r(S) = \begin{cases} p & ; S = 1 \\ 1-p & ; S = -1 \end{cases}$$

WE REPEAT THIS TRIAL n TIMES, SUM THE RESULTS, AND CALL IT $\underline{X}$:

$$\underline{X} = S_1 + S_2 + \ldots + S_n$$

LET US DEFINE $R = \frac{S+1}{2}$. WE THUS HAVE A BERNOULLI TRIAL:

$$P_r[R] = \begin{cases} p & ; R=1 \\ 1-p & ; R=0 \end{cases}$$

CONSIDER THEN

$$\begin{aligned} Y &= R_1 + R_2 + \ldots + R_n \\ &= \frac{S_1+1}{2} + \frac{S_2+1}{2} + \ldots + \frac{S_n+1}{2} \\ &= \frac{1}{2}[S_1 + S_2 + \ldots + S_n] + \frac{n}{2} \\ &= \frac{1}{2}X + n/2 = \frac{X+n}{2} \end{aligned}$$

AS IS SHOWN ON pg. 87 OF THE TEXT, Y IS DISTRIBUTED BINOMIALLY:

$$Y = \frac{X+n}{2} \sim b(n,p)$$

(b) THE MEAN OF A BINOMIAL DISTRIBUTION IS

$$\mu = E[Y] = E\left[\frac{X+n}{2}\right] = np$$

SINCE $E(\cdot)$ IS A LINEAR OPERATOR:

$$\begin{aligned} E\left[\frac{X+n}{2}\right] &= \frac{1}{2}\{E(X+n)\} \\ &= \frac{1}{2}E(X) + \frac{1}{2}n = np \end{aligned}$$

SOLVING FOR $E(X)$ GIVES THE DESIRED RESULT:

$$E(X) = n(2p-1)$$

95/200

Math 5383 Final Exam Dec 14, 1976 Name: Bob Marks

1. Consider the function $f(x) = ke^{-\beta|x-\alpha|}$, $-\infty < x < \infty$; $\beta > 0$, $-\infty < \alpha < \infty$.

(a) Determine k so that $f(x)$ is a p.d.f.

(b) Find the moment generating function of $f(x)$.

(c) Find $E(X)$.

(d) Find $Var(X)$.

(e) Find $F(x)$.

(a) $f(x) = ke^{-\beta|x-\alpha|}$

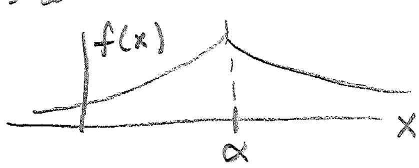
$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} ke^{-\beta|x-\alpha|}\,dx = \int_{-\infty}^{\infty} ke^{-\beta|x|}\,dx = 2k\int_0^{\infty} e^{-\beta x}\,dx = 2k\frac{1}{-\beta}e^{-\beta x}\Big|_0^{\infty} = \frac{2k}{-\beta}[0-1] = \frac{2k}{\beta}$$

$$\Rightarrow k = \frac{\beta}{2}$$

$$f(x) = \frac{\beta}{2}e^{-\beta|x-\alpha|}$$

(b) $M(t) = E[e^{xt}]$

$$= \int_{-\infty}^{\infty} e^{xt}e^{-\beta|x-\alpha|}\,dx$$



FOR $x < \alpha \Rightarrow x-\alpha < 0$

$-\beta|x-\alpha| = \beta(x-\alpha)$

FOR $x > \alpha \Rightarrow x-\alpha > 0$

$-\beta|x-\alpha| = -\beta(x-\alpha)$

$$M(t) = \frac{\beta}{2}\left[\int_{-\infty}^{\alpha} e^{tx}e^{\beta(x-\alpha)}\,dx + \int_{\alpha}^{\infty} e^{tx}e^{-\beta(x-\alpha)}\,dx\right]$$

$$= \left[e^{-\beta\alpha}\int_{-\infty}^{\alpha} e^{x(t+\beta)}\,dx + e^{\beta\alpha}\int_{\alpha}^{\infty} e^{x(t-\beta)}\,dx\right]\frac{\beta}{2}$$

$$= \left[\frac{e^{-\beta\alpha}}{t+\beta}e^{x(t+\beta)}\Big|_{-\infty}^{\alpha} + \frac{e^{\beta\alpha}}{t-\beta}e^{x(t-\beta)}\Big|_{\alpha}^{\infty}\right]\frac{\beta}{2}$$

$$= \frac{\beta}{2}\frac{e^{-\beta\alpha}}{t+\beta}\left[e^{\alpha(t+\beta)} - 0\right] \quad ; t+\beta > 0$$

$$+ \frac{\beta}{2}\frac{e^{\alpha\beta}}{t-\beta}\left[0 - e^{\alpha(t-\beta)}\right] \quad ; t-\beta < 0$$

NOW $t+\beta > 0 \Rightarrow t > -\beta$, $t-\beta < 0 \Rightarrow t < \beta$ $\Rightarrow -\beta < t < \beta$

$$M(t) = \left[\frac{\cancel{e^{-\alpha\beta}}e^{\alpha t}\cancel{e^{\alpha\beta}}}{t+\beta} - \frac{\cancel{e^{\alpha\beta}}e^{\alpha t}\cancel{e^{-\alpha\beta}}}{t-\beta}\right]\frac{\beta}{2}$$

$$= \left(e^{\alpha t}\left[\frac{1}{t+\beta} - \frac{1}{t-\beta}\right]\right)\frac{\beta}{2}$$

$$= \frac{e^{\alpha t}}{t^2-\beta^2}\left[\cancel{t}-\beta-\cancel{t}-\beta\right]\frac{\beta}{2}$$

$$= \frac{-2\beta e^{\alpha t}}{t^2-\beta^2}\left(\frac{\beta}{2}\right); \quad -\beta < t < \beta$$

(c) $E(X) = \frac{d}{dt}M(0)$

$$\cancel{\frac{d}{dt}M(t) = -2\beta\left[\frac{\alpha e^{\alpha t}}{t^2-\beta^2} + 2t\frac{e^{\alpha t}}{(t^2-\beta^2)^2}\right]}$$

$$\cancel{= -2\beta e^{\alpha t}\left[\frac{\alpha(t^2-\beta^2)+2t}{(t^2-\beta^2)^2}\right]}$$

$$\cancel{\frac{d}{dt}M(0) = -2\beta\left[\frac{-\alpha\beta^2}{\beta^4}\right] =}$$

c & d ON BACK

(e) $-\infty < x < \alpha$

$$F(x) = \int_{-\infty}^{x}\frac{\beta}{2}e^{\beta(x-\alpha)}\,dx = \frac{\beta}{2}e^{-\alpha\beta}\int_{-\infty}^{x}e^{\beta x}\,dx = \frac{\beta}{2}e^{-\alpha\beta}\frac{1}{\beta}e^{\beta x} = \frac{1}{2}e^{\beta(x-\alpha)}$$

NOTE: $F(\alpha-) = \frac{1}{2}$

$\alpha < x < \infty$

$$F(x) = \frac{1}{2} + \int_{\alpha}^{x}\frac{\beta}{2}e^{-\beta(x-\alpha)}\,dx$$

$$= \frac{1}{2} + \frac{\beta}{2}e^{\alpha\beta}\int_{\alpha}^{x}e^{-\beta x}\,dx$$

$$= \frac{1}{2} + \frac{\beta}{2}e^{\alpha\beta}\frac{-1}{\beta}\left[e^{-\beta x} - e^{-\alpha\beta}\right]$$

$$= \frac{1}{2} + \frac{1}{2}e^{\alpha\beta}\left[e^{-\alpha\beta} - e^{-\beta x}\right]$$

$$= \frac{1}{2} + \frac{1}{2}\left[1 - e^{-\beta(x-\alpha)}\right]$$

$$= 1 - \frac{1}{2}e^{-\beta(x-\alpha)}$$

OR

$$F(x) = \begin{cases} \frac{1}{2}e^{\beta(x-\alpha)} & ; -\infty < x < \alpha \\ 1 - \frac{1}{2}e^{-\beta(x-\alpha)} & ; \alpha \le x < \infty \end{cases}$$

(c) $M(t) = \dfrac{-2\beta e^{\alpha t}}{t^2 - \beta^2} \cdot \dfrac{\beta}{2}$

$\mu = \dfrac{dM(0)}{dt}$

$\dfrac{dM(t)}{dt} = -2\beta\left[\dfrac{\alpha e^{\alpha t}}{t^2-\beta^2} + e^{\alpha t}\dfrac{d}{dt}\dfrac{1}{t^2-\beta^2}\right]\dfrac{\beta}{2}$

$\dfrac{d}{dt}\dfrac{1}{t^2-\beta^2} = \dfrac{d}{dt}(t^2-\beta^2)^{-1}$

$= -(t^2-\beta^2)^{-2}\,2t = \dfrac{-2t}{(t^2-\beta^2)^2}$

$\Rightarrow \dfrac{dM(t)}{dt} = -2\beta e^{\alpha t}\left[\dfrac{\alpha}{t^2-\beta^2} - \dfrac{2t}{(t^2-\beta^2)^2}\right]\dfrac{\beta}{2}$

$= -2\beta e^{\alpha t}\left[\dfrac{\alpha(t^2-\beta^2) - 2t}{(t^2-\beta^2)^2}\right]\dfrac{\beta}{2}$

$= -2\beta e^{\alpha t}\left[\dfrac{\alpha t^2 - 2t - \alpha\beta^2}{(t^2-\beta^2)^2}\right]\dfrac{\beta}{2}$

$\dfrac{dM(0)}{dt} = -\cancel{2}\beta\left[\dfrac{-\alpha\beta^2}{\beta^4}\right]\dfrac{\beta}{\cancel{2}} = \alpha$

$\Rightarrow \mu = \alpha$

(d) $\overline{X^2} = \dfrac{d^2 M(0)}{dt^2}$

$= \dfrac{d}{dt}\, -\beta^2 e^{\alpha t}\left[\dfrac{\alpha t^2 - 2t - \alpha\beta^2}{(t^2-\beta^2)^2}\right]$

$t^4 - 2\beta^2 t^2 + \beta^4$

$= -\beta^2\Big[\alpha\left\{\dfrac{\alpha t^2 - 2t - \alpha\beta^2}{(t^2-\beta^2)^2}\right\}$

$+ \dfrac{(t^2-\beta^2)^2[2\alpha t - 2] - (4\cancel{t^3} - \cancel{4\beta^2 t})^{\to 0}(\cdot)}{(t^2-\beta^2)^4}\Big]e^{\alpha t}$

$\overline{X^2} = -\beta^2\left[\alpha\left(\dfrac{-\alpha\beta^2}{\beta^4}\right) + \dfrac{\beta^4(-2)}{\beta^8}\right]$

$= -\beta^2\left[\dfrac{-\alpha^2}{\beta^2} - \dfrac{2}{\beta^4}\right]$

$= \alpha^2 + \dfrac{2}{\beta^2}$

$\operatorname{var} X = \overline{X^2} - \overline{X}^2$

$= \left[\alpha^2 + \dfrac{2}{\beta^2}\right] - \alpha^2$

$= 2/\beta^2$

2. Find the probability that a poker hand (5 cards) drawn from a deck of 52 cards contains

(a) 2 aces, 1 jack, and 2 queens;

(b) at least one diamond;

(c) either 4 aces, or 3 aces and 2 kings.

$\frac{36}{4}$ / 144 $\quad$ $\frac{52}{13}$ / 39

(a) $$\frac{\binom{4}{2}\binom{4}{1}\binom{4}{2}\binom{40}{0}}{\binom{52}{5}} = \frac{4\binom{4}{2}^2}{\binom{52}{5}} = \frac{4\left(\frac{4\cdot 3}{2}\right)^2}{\binom{52}{5}} = \frac{144}{\binom{52}{5}}$$

(b) P[AT LEAST 1 DIAMOND] = 1 - P[NO DIAMAND]

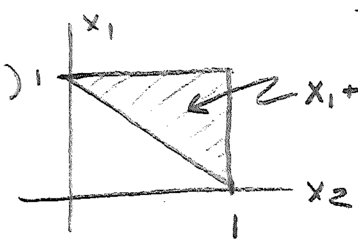
$$= 1 - \frac{\binom{39}{5}\binom{13}{0}}{\binom{52}{5}} = 1 - \frac{\binom{39}{5}}{\binom{52}{5}}$$

(c) THESE ARE MUTUALLY EXCLUSIVE EVENTS

$$\Rightarrow P_r = \frac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} + \frac{\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$$

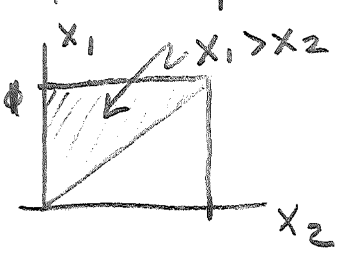
$$= \frac{48 + 4\binom{4}{2}}{\binom{52}{5}}$$

3. Let X_1, X_2 denote a random sample of size 2 from a distribution which is uniform over the interval $(0,1)$. Compute (a) $P_r(X_1 + X_2 \geq 1)$, (b) $P_r(X_1 > X_2)$, (c) $P_r(X_1 + X_2 \geq 1 / X_1 > \frac{1}{2})$.

(a) 

$x_1 + x_2 \geq 1$

$$P_r[x_1 + x_2 \geq 1] = \int_{①} dx_1 dx_2 = \frac{1}{2}$$

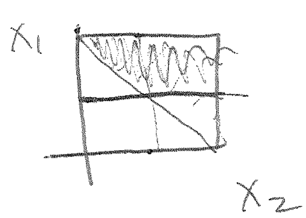
(b) 

$x_1 > x_2$

$$P_r[x_1 > x_2] = \int_{①} dx_1 dx_2 = \frac{1}{2}$$

(c)
$$P_r[x_1 + x_2 \geq 1 / x_1 > \tfrac{1}{2}] = P_r[x_2 \geq \tfrac{1}{2}] = \cancel{\tfrac{1}{2}}$$

-3



4. Let X_1, X_2 denote a random sample from a normal distribution $n(\cancel{\mu}^{0}, \sigma^2)$. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 + kX_2$.

(a) Find the joint p.d.f., $g(y_1, y_2)$, of Y_1 and Y_2 by the change of variable technique.

3 pts (b) Find $g(y_1, y_2)$ by the moment generating function technique.

(c) What "type" of distribution is $g(y_1, y_2)$? JUST STATE

(d) Are Y_1 and Y_2 independent for all values of k?

(e) If not find a value of k such that Y_1 and Y_2 are independent.

(a) $Y_1 = X_1 + X_2 \qquad Y_2 = X_1 + kX_2$

$Y_2 = X_1 + kX_2 \qquad kY_1 = kX_1 + kX_2$

$Y_1 - Y_2 = X_2 - kX_2 = X_2(1-k) \qquad Y_2 - kY_1 = X_1 - kX_1 = X_1(1-k)$

$\Rightarrow X_2 = \frac{Y_2 - Y_1}{K-1} \qquad \Rightarrow X_1 = \frac{KY_1 - Y_2}{K-1}$

$$J = \begin{vmatrix} \frac{K}{K-1} & \frac{-1}{K-1} \\ \frac{-1}{K-1} & \frac{1}{K-1} \end{vmatrix} = \left| \frac{K}{(K-1)^2} - \frac{1}{(K-1)^2} \right| = \frac{1}{K-1}$$

$$f(x_1, x_2) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x_1^2 + x_2^2)}{2\sigma^2}}$$

$$g(Y_1, Y_2) = J\, f(X_1(Y), X_2(Y))$$

$$= \frac{1}{|K-1|} \frac{1}{2\pi\sigma^2} e^{-\frac{(KY_1 - Y_2)^2 + (Y_2 - Y_1)^2}{2(K-1)^2\sigma^2}}$$

EXPANDING NUMERATOR:

$K^2Y_1^2 - 2KY_1Y_2 + Y_2^2 + Y_2^2 - 2Y_1Y_2 + Y_1^2$

$(K^2+1)Y_1^2 - 2(K+1)Y_1Y_2 + 2Y_2^2$

$$(*) \Rightarrow g(Y_1, Y_2) = \frac{1}{|K-1|\, 2\pi\sigma^2} e^{\frac{-[(K^2+1)Y_1^2 - 2(K+1)Y_1Y_2 + 2Y_2^2]}{2(K-1)^2\sigma^2}}$$

(b) $M_X(t_1, t_2) = e^{\frac{1}{2}\sigma^2[t_1^2 + t_2^2]}$

$= E\left[e^{Y_1t_1 + Y_2t_2}\right] = E\left[e^{(t_1+t_2)X_1} \ldots \text{etc.}\right]$

$= E\left[e^{\frac{KY_1 - Y_2}{K-1}t_1 + \frac{Y_2 - Y_1}{K-1}t_2}\right]$

ETC.

(DON'T REMEMBER MGF FOR BIVARIATE NORMAL)

(c) FROM (*), THE DISTRIBUTION IS OBVIOUSLY BIVARIATE NORMAL

(d) NO! THIS TERM GIVES A CORRELATION COEFFICIENT. IF THERE'S A $\rho \neq 0$, THEN THERE'S NOT INDEPENDENCE

(e) WE CAN MAKE THIS TERM VANISH BY CHOOSING $K = -1$. IF A BIVARIANT NORMAL HAS $\rho = 0$, THEN Y_1 AND Y_2 ARE INDEPENDENT (THIS CAN ALSO BE SEEN FROM THE RESULTING SEPARABILITY OF $g(Y_1, Y_2)$)

5. Let X be a continuous random variable. Suppose $X_1, \ldots, X_n$ is a random sample from $f(x)$. Find the p.d.f. of (a) $Y = \min(X_1, X_2, \ldots, X_n)$, (b) $Y = \max(X_1, X_2, \ldots, X_n)$.

(a) $\cancel{F_Y(Y) = P_r[Y \leq Y] = 1 - P_r[Y > Y] = 1 - P_r[Y \geq Y]}$

$\cancel{= 1 - P_r[X_1 \geq Y, X_2 \geq Y, \ldots, X_n \geq Y]}$ LET $X_i \sim f_X(x)$, $i = 1, \ldots, n$

$\cancel{= 1 - (P_r[X \geq Y])^n = 1 - \left[\int_{-\infty}^{x} f_X(x)\, dx\right]^n}$

OR, IF WE DEFINE DISTRIBUTION: $\cancel{F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx}$

$\cancel{\Rightarrow F_Y(Y) = 1 - F_X^n(Y) \Rightarrow f_Y(Y) = \frac{dF_Y(Y)}{dY} = -\frac{d}{dY}F_X^n(Y) = -nF_X^{n-1}(Y)f_X(Y)}$

OVER

(a) $Y = \min(x_1, x_2, \ldots, X_n)$

$$\Pr[Y \geq y] = \Pr[X_1 \geq y; X_2 \geq y, \ldots X_n \geq y]$$
$$= \Pr[X_1 \geq y] \Pr[X_2 \geq y] \cdots \Pr[X_n \geq y]$$

$$F_Y(y) = 1 - \Pr[Y \geq y] \quad \text{(for continuous)}$$
$$= 1 - \{\Pr[X \geq y]\}^n$$

DENOTE DISTRIBUTION OF X AS $F_X(x) = \Pr[X \leq x]$

$$\Rightarrow F_Y(y) = 1 - [1 - F_X(y)]^n$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = -\frac{d}{dy}[1 - F_X(y)]^n$$
$$= -n[1 - F_X(y)]^{n-1} \frac{d}{dy}[1 - F_X(y)]$$
$$= n f_X(y) [1 - F_X(y)]^{n-1}$$

WHERE

$$f_X(x) = \frac{d}{dx} F_X(x) = \text{PDF OF } X$$

(b) $Y = \max(x_1, \ldots X_n)$

$$\underbrace{\Pr[Y \leq y]}_{F_Y(y)} = [\underbrace{\Pr\{X \leq y\}}_{F_X(y)}]^n \quad ; \quad X = X_i, \; i = 1, \ldots, n$$

$$\Longrightarrow F_Y(y) = F_X^n(y)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$
$$= n F_X^{n-1}(y) \frac{d}{dy} F_X(y)$$
$$= n \underbrace{f_X(y)}_{\text{PDF OF } X} F_X^{n-1}(y)$$

6. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a normal distribution, $n(\mu, \sigma^2)$. Let $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ and $S_n^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}$. (a) Find $E(S_n^2)$ and $Var(S_n^2)$.

(b) Show that S_n^2 converges to σ^2 in probability. (you may assume the distribution of $\frac{nS_n^2}{\sigma^2}$ to work the problem.) $\chi^2(n-1)$

$\chi_r^2 \to f(x) = \frac{1}{\Gamma(\frac{r}{2})2^{r/2}} X^{\frac{r}{2}-1} e^{-x/2}$ $\quad M(t) = (1-2t)^{-r/2}$, $\mu = r$ $\quad \sigma^2 = 2r$

NOW $\frac{nS_n^2}{\sigma^2} \sim \chi^2_{n-1}(x) \Rightarrow S_n^2 \sim \frac{\sigma^2}{n}\chi^2_{n-1}\left[\frac{\sigma^2 x}{n}\right]$

IF $Y \sim \chi^2_{n-1}$, THEN $E[Y] = n-1$ and $Var[Y] = 2(n-1)$

$\Rightarrow E\left[\frac{nS_n^2}{\sigma^2}\right] = n-1 = \frac{n}{\sigma^2}E[S_n^2] \Rightarrow E(S_n^2) = \frac{\sigma^2(n-1)}{n}$

$Var\left[\frac{nS_n^2}{\sigma^2}\right] = 2(n-1) = \frac{n^2}{\sigma^4} Var\, S_n^2 \Rightarrow Var(S_n^2) = \frac{2\sigma^4(n-1)}{n^2}$

(b) $\lim_{n\to\infty} P_r[|S_n^2 - \sigma^2| < \epsilon] = 1 \Leftarrow$ WE MUST SHOW THIS

CHEBYCHEV: $P_r[|X - \mu| \geq k] \leq \frac{Var\, X}{k^2}$ $\quad X = S_n^2$, $\mu = \sigma^2(n-1)/n$, $Var\, X = \frac{2\sigma^4(n-1)}{n^2}$

$\Rightarrow P_r\left[\left|S_n^2 - \frac{\sigma^2(n-1)}{n}\right| \geq k\right] \leq \frac{2\sigma^4(n-1)}{k^2 n^2} \geq 0$

AS $n \to \infty$, $\frac{n-1}{n} \to 1$ AND, FOR ANY FIXED $k > 0$, $\frac{2\sigma^2(n-1)}{k^2 n^2} \to 0$

THUS, AS $n \to \infty$, $P_r[|S_n^2 - \sigma^2| \geq k] \leq 0 \geq 0 \Rightarrow P_r[|S_n^2 - \sigma^2| \geq k] = 0 \ \forall k > 0$

IN THE LIMIT $\Rightarrow P_r[|S_n^2 - \sigma^2| < \epsilon] = 1 - P_r[|S_n^2 - \sigma^2| \geq \epsilon] = 1 - 0 = 1$

(k & ε ARE THE SAME CREATURES) QED

GOOD only for small p

7. Let Y_n have a $b(n, p)$ distribution. Assume $E(Y_n) = \mu = np$ is the same for every n; i.e. $p = \mu/n$ where μ is a constant. (a) Find the limiting distribution of Y_n.

(b) Let Y have a $b(1000, .005)$ distribution. Find $Pr(Y \leq 2)$. (Note: $e^{-5} = .0067$). $\to M(t) = [(1-p)+pe^t]^n$

Pois $\to M(t) = e^{\mu(e^t - 1)}$

(a) $M_n(t) = [(1-p) + pe^t]^n$
$= [(1 - \frac{\mu}{n}) + \frac{\mu}{n}e^t]^n$

$\mu^x =$

$\ln M_n(t) = \ln[(1 - \frac{\mu}{n}) + \frac{\mu}{n}e^t] / \frac{1}{n}$

(*) $\lim_{n\to\infty} \ln M_n(t) = \lim_{n\to\infty} \frac{\left(\frac{+\frac{\mu}{n^2} - \frac{\mu}{n^2}e^t}{1 - \frac{\mu}{n} + \frac{\mu}{n}e^t}\right)}{-1/n^2} \Rightarrow \frac{-n^2 \frac{\mu}{n^2}(1-e^t)}{1 - \frac{\mu}{n} + \frac{\mu}{n}e^t}$

$= -\mu(1 - e^t)$

$\Rightarrow \lim_{n\to\infty} M_n(t) = e^{+\mu(e^t - 1)} \leftarrow$ POISSON M.G.F.

(b) $Y_{1000} \sim$ POISSON $(\mu = np = (1000)(0.005) = 5) = \frac{5^y e^{-5}}{y!}$

$P_r[Y \leq 2] = P_r[Y = 0] + P_r[Y = 1] + P_r[Y = 2]$

$= e^{-5}\left[\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!}\right] = e^{-5}[1 + 5 + 12.5] = e^{-5}\, 18.5$

$= 0.1044$

*HERE, WE HAVE TREATED n AS A CONTINUOUS PARAMETER (WITHOUT LOSS OF GENERALITY) & APPLIED LA'HOPITAL'S RULE (FORGOT HOW BOOK DID IT)

BOB MARKS
1/20/77

P: LET X_1, X_2 DENOTE A RANDOM SAMPLE FROM A NORMAL $n(0,\sigma^2)$. LET $Y_1 = X_1 + X_2$ AND $Y_2 = X_1 + kX_2$.

(a) FIND THE JOINT p.d.f., $g(Y_1, Y_2)$, OF Y_1 AND Y_2 BY THE CHANGE OF VARIABLE TECHNIQUE.

(b) FIND $g(Y_1, Y_2)$ BY THE MOMENT GENERATING FUNCTION TECHNIQUE.

(c) WHAT "TYPE" OF DISTRIBUTION IS $g(Y_1, Y_2)$?

(d) ARE Y_1 AND Y_2 INDEPENDENT FOR ALL VALUES OF k?

(e) IF NOT, FIND A VALUE OF k SUCH THAT Y_1 AND Y_2 ARE INDEPENDENT.

(a) $Y_1 = X_1 + X_2 \qquad Y_2 = X_1 + K X_2$

$Y_1 - Y_2 = X_2 - K X_2 \Rightarrow X_2 = \frac{Y_1 - Y_2}{-(K-1)}$

$K Y_1 - Y_2 = K X_1 - X_1 \Rightarrow X_1 = \frac{K Y_1 - Y_2}{K-1}$

$$J = \begin{bmatrix} \delta X_1/\delta Y_1 & \delta X_1/\delta Y_2 \\ \delta X_2/\delta Y_1 & \delta X_2/\delta Y_2 \end{bmatrix} = \begin{bmatrix} \frac{K}{K-1} & \frac{-1}{K-1} \\ \frac{-1}{K-1} & \frac{1}{K-1} \end{bmatrix}$$

$|J| = \frac{1}{|K-1|}$

NOW

$$f(x_1 x_2) = \frac{1}{2\pi\sigma^2} e^{-\left[\frac{x_1^2 + x_2^2}{2\sigma^2}\right]}$$

AND

$$g(Y_1 Y_2) = |J| f\left[X_1(Y_1 Y_2), X_2(Y_1 Y_2)\right]$$

$$= \frac{1}{2\pi|K-1|\sigma^2} e^{-\frac{1}{2}\frac{1}{(K-1)^2\sigma^2}\left[(KY_1 - Y_2)^2 + (Y_1 - Y_2)^2\right]}$$

$$= \frac{1}{2\pi|K-1|\sigma^2} e^{-\frac{1}{2}\frac{1}{(K-1)^2\sigma^2}\left[(K^2+1)Y_1^2 - 2(K+1)Y_1Y_2 + 2Y_2^2\right]} \quad (1)$$

(c) THIS LOOKS LIKE IT MIGHT BE A BIVARIATE NORMAL DISTRIBUTION, THE GENERAL FORM OF WHICH IS (p. 111)

$$g_b(Y_1 Y_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}\frac{1}{1-\rho^2}\left[\left(\frac{Y_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{(Y_1-\mu_1)(Y_2-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{Y_2-\mu_2}{\sigma_2}\right)^2\right]} \quad (2)$$

NOW $X_i \sim n(0, \sigma^2) \quad ; i = 1, 2$

FROM THEOREM I ON p. 158

$Y_1 \sim n(0, 2\sigma^2)$

$Y_2 \sim n(0, (K^2+1)\sigma^2)$

THUS $\mu_1 = \mu_2 = 0$

AND $\sigma_1^2 = 2 \;, \sigma_2^2 = (K^2+1)\sigma^2$

SUBSTITUTING INTO (2) $\Longrightarrow$

$$g_b(Y_1, Y_2) = \frac{1}{2\pi \sqrt{2}\sqrt{K^2+1}\, \sigma^2 \sqrt{1-\rho^2}} \exp -\frac{1}{2} \frac{1}{1-\rho^2} \Bigg[\frac{Y_1^2}{2} - 2\rho \frac{Y_1 Y_2}{\sqrt{2(K^2+1)}\,\sigma^2} + \frac{Y_2^2}{(K^2+1)\sigma^2} \Bigg] \qquad (3)$$

COMPARING WITH (3), WE MUST HAVE (FROM THE EXPONENTIAL'S COEFFICIENT)

$$\sqrt{2(K^2+1)}\, \sigma^2 \sqrt{1-\rho^2} = |K-1|\, \sigma^2$$

OR

$$\sqrt{1-\rho^2} = \frac{|K-1|}{\sqrt{2(K^2+1)}}$$

OR

$$1-\rho^2 = \frac{(K-1)^2}{2(K^2+1)} \qquad (4)$$

$$\rho = \left[1 - \frac{(K-1)^2}{2(K^2+1)}\right]^{\frac{1}{2}} \qquad (5)$$

SUBSTITUTING (4) AND (5) INTO THE EXPONENT OF (3) MUST RESULT IN THE EXPONENT OF (2) IF $g(Y_1, Y_2)$ IS BIVARIATE NORMAL:

$$\frac{2(K^2+1)}{(K-1)^2} \Bigg[\frac{Y_1^2}{2} - 2\left\{1 - \frac{(K-1)^2}{2(K^2+1)}\right\}^{\frac{1}{2}} \frac{Y_1 Y_2}{\sqrt{2(K^2+1)}\,\sigma^2} + \frac{Y_2^2}{(K^2+1)\sigma^2}$$

$$= \frac{1}{\sigma^2 (K-1)^2} \Bigg[(K^2+1) Y_1^2 - 2\left\{2(K^2+1)\right\} \left\{ \frac{1}{2(K^2+1)} - \frac{(K-1)^2}{2(K^2+1)^2} \right\}^{\frac{1}{2}} Y_1 Y_2 + 2Y_2^2 \Bigg]$$

$$= \frac{1}{\sigma^2(K-1)^2}\Big[(K^2+1)Y_1^2 - 2\big[(2K^2+2)-(K^2-2K+1)\big]^{\frac{1}{2}} Y_1 Y_2 + 2Y_2^2\Big]$$

$$= \frac{1}{\sigma^2(K-1)^2}\Big[(K^2+1)Y_1^2 - 2\big[(K+1)^2\big]^{\frac{1}{2}} Y_1 Y_2 + 2Y_2^2\Big]$$

$$= \frac{1}{\sigma^2(K-1)^2}\Big[(K^2+1)Y_1^2 - 2(K+1)Y_1Y_2 + 2Y_2^2\Big]$$

THIS, COMPARES FAVORABLY WITH THE EXPONENT IN ②. THUS, $g(Y_1Y_2)$ IS BIVARIATE NORMAL WITH

$$\rho = {}^{+}\left[1 - \frac{(K-1)^2}{2(K^2+1)}\right]^{\frac{1}{2}} \qquad ⑥$$

$$(b)\ M(t_1, t_2) = E\left[e^{t_1 Y_1 + t_2 Y_2}\right]$$
$$= E\left[e^{t_1(X_1 + X_2) + t_2(X_1 + kX_2)}\right]$$
$$= E\left[e^{(t_1+t_2)X_1}\, e^{(t_1 + kt_2)X_2}\right]$$

BUT X_1 AND X_2 ARE INDEPENDENT $n(0, \sigma^2]$

$$\Rightarrow M(t_1 t_2) = E\left[e^{(t_1+t_2)X_1}\right] E\left[e^{(t_1+kt_2)X_2}\right]$$

$$= e^{\frac{\sigma^2}{2}(t_1+t_2)^2}\, e^{\frac{\sigma^2}{2}(t_1+kt_2)^2}$$

$$= e^{\frac{\sigma^2}{2}\left[2t_1^2 + (k^2+1)t_2^2 + 2(k+1)t_1 t_2\right]}$$

$$= e^{\frac{1}{2}\left[2\sigma^2 t_1^2 + (k^2+1)\sigma^2 t_2^2 + \sigma^2 2(k+1) t_1 t_2\right]}$$

$$= e^{\frac{1}{2}\left[(2\sigma^2)t_1^2 + 2\{\frac{1}{2}\sigma^2 2(k+1)\} t_1 t_2 + (k^2+1)\sigma^2 t_2^2\right]} \quad (7)$$

FROM p. 114, THIS IS THE MGF OF A BIVARIATE NORMAL DISTRIBUTION WITH

$$\sigma_1^2 = 2\sigma^2 \quad (8)$$
$$\sigma_2^2 = (k^2+1)\sigma^2 \quad (9)$$
$$\rho\sigma_1\sigma_2 = \sigma^2(2k+1) \quad (10)$$

SUBSTITUTING (8) & (9) INTO (10)

$$\rho\left[\sqrt{2(k^2+1)}\right] = (k+1)$$
$$\Rightarrow \rho^2 = \frac{(k+1)^2}{2(k^2+1)}$$

$$= \left[1 - \frac{(k-1)^2}{2(k^2+1)}\right] \quad (11)$$

NOT SURPRISINGLY, THIS IS THE SAME ANSWER WE GOT IN THE PREVIOUS SECTION.

(d) Y_1 & Y_2 ARE NOT INDEPENDENT FOR ALL VALUES OR K. USING THM. 3 ON p. 114, WE MAY USE A VALUE OF $K=1$ IN (11) TO GIVE $\rho^2 = 1$ AT WHICH TIME Y_1 & Y_2 ARE COMPLETELY CORRELATED, AND THUS DEPENDENT

(e) AGAIN USING THIS THEM, A VALUE OF $K=-1$ WILL GIVE $\rho = 0$ ASSURING STATISTICAL INDEPENDENCE OF Y_1 & Y_2.

1. Discuss sufficient statistics for

(a) $f(x;\theta) = \theta(1-\theta)^{x-1}$, $x = 1, 2, \ldots$, $0 < \theta < 1$

(b) uniform p.d.f. $(0, \theta)$

(c) uniform p.d.f. (θ_1, θ_2)

(d) $f(x;\theta) = \theta e^{-\theta x}$, $x > 0$, $\theta > 0$.

(e) $f(x;\theta) = \dfrac{\theta^r}{\Gamma(r)} x^{r-1} e^{-\theta x}$, $x > 0$.

2. Let $f(x;\theta) = \theta x^{\theta - 1}$, $0 < x < 1$. Find a confidence interval for θ. (What is the p.d.f. of $F(x;\theta)$? What is the p.d.f. of $-\ln F(x;\theta)$?)

3. Find a confidence interval for θ where $f(x;\theta) = \frac{1}{\theta}$, $0 < x < \theta$.

(CENTRAL LIMIT THM)

4. Given any p.d.f. with $\sigma_x^2 < \infty$. Discuss the construction of a C.I. for $\mu = E(X)$

T:

5. Is there an unbiased statistic for $\frac{1}{\theta}$ if $f(x;\theta) = \frac{\theta^x e^{-\theta}}{x!}$, $\theta > 0$, $x = 0, 1, 2, \ldots$

6. Let $f(x;\theta) = \frac{1}{\Gamma(p)}(x-\theta)^{p-1} e^{-(x-\theta)}$. Is there a sufficient statistic for θ?

7. Suppose Y is sufficient for θ. Any 1 to 1 function of Y is also sufficient. What about functions which are not 1 to 1?

8. Find a large sample confidence interval for θ, the parameter in a Poisson p.d.f.

9. Show that for the p.d.f. $f(x;\theta) = \frac{1}{\theta}$, $0 < x < \theta$ and confidence coefficient $1-\alpha$, confidence limits for θ are t and t/ψ, where t is the sample range and ψ is given by
$$\psi^{n-1}\{n-(n-1)\psi\} = \alpha.$$

sample range $= \max(X_1, \ldots, X_n) - \min(X_1, \ldots, X_n)$

10. Consider $f(x;\theta) = g(x)/h(\theta)$, $a(\theta) \le x \le b(\theta)$, for $b(\theta)$ is a monotone decreasing function of $a(\theta)$, show that $Y_1 = \min(X_i)$ and $Y_n = \max(X_i)$ are a pair of joint sufficient statistics for θ. Find the single sufficient statistic for θ.

Show that $\psi = \dfrac{h(\hat{\theta})}{h(\theta)}$ has p.d.f. $f(\psi) = n\psi^{n-1}$, $0 \le \psi \le 1$

Show that $P\{\alpha^{\frac{1}{n}} \le \psi \le 1\} = 1-\alpha$ and set a confidence interval on θ. Show that this interval is shorter than ~~the one~~ any other interval based on the p.d.f. of ψ.

Use the above to show that a C.I. for θ in $f(x;\theta) = \frac{1}{\theta}$ $0 \le x \le \theta$ is obtainable from $P\{\max(X_i) \le \theta \le \max(X_i)\,\alpha^{-1/n}\} = 1-\alpha$ and that this is shorter than the interval in problem 9.

1. DISCUSS SUFFICIENT STATISTICS FOR

(a) $f(x;\theta) = \theta(1-\theta)^{x-1}$; $x = 1, 2, \dots$ $0<\theta<1$

$= \frac{\theta}{(1-\theta)}(1-\theta)^x$

$$\prod_{i=1}^{n} f(x_i;\theta) = \underbrace{\left[\frac{\theta}{1-\theta}\right]^n (1-\theta)^{\Sigma x_i}}_{K_1} \times \underbrace{1}_{K_2}$$

THUS, $Y = \sum^{n} x_i$ IS A SUFF. STATISTIC FOR θ.

(b) $f(x;\theta) = \frac{1}{\theta}$; $0 < x < \theta$

LET $Y = MAX(x_1 \dots x_n)$

$P_r[Y \le y] = [P_r[X \le x]]^n$

$F_Y(Y) = F_X^n(Y) \Rightarrow f_Y(Y) = n f_X(Y) F_X^{n-1}(Y)$

$F_X(Y) = \frac{x}{\theta} \Rightarrow f_Y(Y) = n \frac{1}{\theta} \frac{x^{n-1}}{\theta}$

$= \frac{n}{\theta^n} Y^{n-1}$; $0 < x < \theta$

NOW $\prod_{i=1}^{n} f(x;\theta) = \frac{1}{\theta^n}$; $0 < x_i < \theta$

$$= \underbrace{\left[\frac{n}{\theta^n} Y^{n-1}\right]}_{g(Y;\theta)} \underbrace{\left[1/n Y^{n-1}\right]}_{H(x_1 \dots x_n)}$$

$\Rightarrow$ Y IS SUFF FOR θ

(c) $f(x;\theta_1,\theta_2) = \frac{1}{\theta_2 - \theta_1}$ $\theta_1 < x < \theta_2$

$Y_1 = \min x_i \Rightarrow 1 - F_Y(Y_1) = [1 - F_Y(x)]^n$

$F(Y_1) = 1 - [1 - F_Y(x)]^n$

$f(Y_1) = n f_Y(x)[1 - F_Y(Y)]^{n-1}$

$Y_n = \max x_i$

WE MUST FIND

$f(Y_1, Y_n)$

$F(Y_1, Y_n) = P_r[Y_1 \le Y_1, Y_n \le Y_n]$

IN GENERAL

$f(Y_1, Y_2, \dots Y_n) = n! f(Y_1) \dots f(Y_n)$

$\Rightarrow f(Y_1, Y_n) = n! f(Y_1) f(Y_n)$ $\Rightarrow$

$$f(Y_1) = n f_X(Y)\left[1 - F_X(Y)\right]^{n-1}$$
$$f(Y_n) = n f_X(Y) F_X^{n-1}(Y)$$
$$\Rightarrow f(Y_1, Y_n) = n^2 n!\ f_X(Y)^2 \left[F_X(Y)(1 - F_X(Y))\right]^{n-1}$$

NOW $\quad f_X(Y) = \dfrac{1}{\theta_2 - \theta_1} \quad ; \theta_1 < Y < \theta_2$

$\quad F_X(Y) = \dfrac{Y - \theta_1}{\theta_2 - \theta_1} \quad ; \theta_1 < Y < \theta_2$

$$\Rightarrow f(Y_1, Y_n) = n^2 n! \frac{1}{(\theta_2 - \theta_1)^2}\left[\left(\frac{Y - \theta_1}{\theta_2 - \theta_1}\right)\left[1 - \frac{Y - \theta_1}{\theta_2 - \theta_1}\right]\right]^{n-1}$$
$$= \frac{n^2 n!}{(\theta_2 - \theta_1)^2}\left[\left(\frac{Y - \theta_1}{\theta_2 - \theta_1}\right)\frac{\theta_2 - Y}{\theta_2 - \theta_1}\right]^{n-1}$$
$$= n^2 n! \frac{1}{(\theta_2 - \theta_1)^{n+1}}\left[(Y - \theta_1)(\theta_2 - Y)\right]^{n-1}$$

NOW
$$\prod^n f(x_i; \theta_1; \theta_2) = \frac{1}{(\theta_2 - \theta_1)^n} \quad \text{(ARG!)}$$

(d) $f(x; \theta) = \theta e^{-\theta x} \quad ; x > 0 \quad \theta > 0$
$$\prod_{i=1}^n f(x_i; \theta) = \theta^n e^{-\theta \Sigma^n x}$$
$$\Longrightarrow Y = \sum^n X \text{ IS SUFFICIENT}$$

(e) $f(x; \theta) = \dfrac{\theta^r}{\Gamma(r)} x^{r-1} e^{-\theta x} \quad ; x > 0$

$$\prod^n f(x_i; \theta) = \underbrace{\left[\frac{\theta^r}{\Gamma(r)}\right]^n e^{-\theta \Sigma x}}_{K_1} \underbrace{\prod_{i=1}^n x_i^{r-1}}_{K_2}$$

$Y = \Sigma x_i$ IS SUFFICIENT

2. $f(x;\theta) = \theta\, x^{\theta-1}$; $0 < x < 1$; $\theta > 0$

$$Y = -\theta \ln X \Rightarrow X = e^{-Y/\theta} \Rightarrow \left|\frac{dX}{dY}\right| = \frac{1}{\theta} e^{-Y/\theta}$$

$$g(Y;\theta) = \left[\frac{1}{\theta} e^{-Y/\theta}\right]\left[\theta \left\{e^{-Y/\theta}\right\}^{\theta-1}\right] \quad ; 0 < Y < \infty$$

$$= e^{-Y/\theta}\, e^{-Y}\, e^{Y/\theta}$$

$$= e^{-Y} \quad ; 0 < Y < \infty$$

$$\int_0^a e^{-Y} dY = \frac{1-\alpha}{2} = 1 - e^{-a} \Rightarrow e^{-a} = 1 - \frac{1}{2} + \frac{\alpha}{2} = \frac{1+\alpha}{2}$$

$$a = -\ln \frac{1+\alpha}{2} > 0$$

$$\int_b^\infty e^{-Y} dY = \frac{1-\alpha}{2} = e^{-b} \Rightarrow b = -\ln \frac{1-\alpha}{2} > 0$$

$$\Rightarrow P_r[a < Y < b] = \alpha$$

$$= P_r[a < -\theta \ln X < b]$$

$$= P_r\left[a < \theta \ln \frac{1}{X} < b\right] \quad ; \ln \frac{1}{X} > 0$$

GIVES CONFIDENCE:

$$\alpha = P_r\left[\frac{a}{\ln \frac{1}{X}} < \theta < \frac{b}{\ln \frac{1}{X}}\right]$$

$$= P_r\left[\frac{\ln \frac{2}{1+\alpha}}{\ln 1/X} < \theta < \frac{\ln \frac{2}{1-\alpha}}{\ln 1/X}\right]$$

3. $f(x;\theta) = \frac{1}{\theta} \qquad ; 0 < x < \theta$

LET $Y_n = \max(x_1 \dots x_n)$

$$f_Y(Y) = n\, f_x(Y) \left[F_x(x) \right]^{n-1}$$
$$= n \frac{1}{\theta} \left[\frac{x}{\theta} \right]^{n-1}$$
$$= n \frac{1}{\theta^n} Y^{n-1} \qquad ; 0 < Y < \theta$$

$$P_r[Y_n \le Y] = \left(\frac{Y}{\theta}\right)^n$$
$$P_r[Y_n \le \theta] = 1$$
$$P_r[Y_n \ge Y] = 1 - \left(\frac{Y}{\theta}\right)^n$$
$$\Rightarrow P_r[Y \le Y_n \le \theta] = 1 - \left(\frac{Y}{\theta}\right)^n$$
$$F_{Y_n}(Y) = \left(\frac{Y}{\theta}\right)^n \qquad ; 0 < Y < \theta$$

LET $Z = \left(\frac{Y}{\theta}\right)^n \qquad ; 0 < Z < 1$

$$Y = \theta Z^{1/n} ; \frac{dY}{dZ} = \frac{\theta}{n} Z^{\frac{1}{n}-1}$$

$$f(z;\theta) = \frac{n}{\theta^n} \frac{\theta}{n} Z^{\frac{1}{n}-1} \left[\theta Z^{\frac{1}{n}}\right]^{n-1}$$
$$= Z^{\frac{1}{n}-1}\, Z\, Z^{-\frac{1}{n}}$$
$$= 1 \qquad ; 0 < Z < 1$$

$$\alpha = b - a = P_r[a < Z < b] = P_r\left[a < \left(\frac{Y}{\theta}\right)^n < b\right]$$
$$= P_r\left[b < \left(\frac{\theta}{Y}\right)^n < a\right]$$
$$= P_r\left[Y \sqrt[n]{b} < \theta < Y \sqrt[n]{a}\right]$$

5. $f(x;\theta) = \dfrac{\theta^{x} e^{-\theta}}{x!}$

$Y = \Sigma^{n} x_i$

$$\Rightarrow g(Y;\theta) = \frac{(n\theta)^{Y} e^{-n\theta}}{Y!}$$

$$E[Y] = e^{-n\theta} \sum_{Y=0}^{\infty} \frac{Y(n\theta)^{Y}}{Y!}$$

$$= e^{-n\theta} \sum_{Y=0}^{\infty} \frac{(n\theta)^{Y}}{(Y-1)!}$$

$$= e^{-n\theta} \sum_{Y=1}^{\infty} \frac{(n\theta)^{Y}}{(Y-1)!}$$

$$\hat{Y} = Y - 1 \Rightarrow Y = \hat{Y} + 1$$

$$E[Y] = e^{-n\theta} \sum_{\hat{Y}=0}^{\infty} \frac{(n\theta)^{\hat{Y}+1}}{\hat{Y}!}$$

$$= e^{-n\theta} (n\theta) \sum_{\hat{Y}=0}^{\infty} \frac{(n\theta)^{Y}}{Y!}$$

$$= e^{-n\theta} (n\theta) e^{n\theta} = n\theta$$

$\Rightarrow Y/n = \bar{X}$ IS AN UNBIASED STATISTIC FOR θ

<u>BUT</u>

WHAT ABOUT $\dfrac{1}{\theta}$ $\Longrightarrow$ ARC

[illegible]

1. Let $X_1, X_2, \ldots, X_n$ be a sample from a $n(\mu, \sigma^2)$, where μ is <u>known</u>.

(a) Determine a confidence interval for σ^2. Describe the limits completely.

(b) Same as (a) except μ is <u>unknown</u>.

(c) A 95% confidence interval for σ^2 is $\left(\frac{nS^2}{a}, \frac{nS^2}{b}\right)$ where a and b are chosen so that $\int_a^b f(x)\,dx = .95$ where $f(x)$ is a χ^2-p.d.f. with $n-1$ d.f. Find a relationship between a and b so that the length of the confidence interval is minimized subject to the condition $\int_a^b f(x)\,dx = .95$.

(d) Find the expected length of the interval in (a) and (b).

2. Let $X \sim n(\mu_1, \sigma_1^2)$ and $Y \sim n(\mu_2, \sigma_2^2)$. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ denote samples from the distributions of X and Y, respectively. Suppose μ_1 is <u>known</u> and μ_2 is <u>unknown</u>. Determine a confidence interval for σ_2^2/σ_1^2 with confidence coefficient …

3. Let $X_1, X_2, \ldots, X_n$ denote a sample from a $n(\theta, \sigma^2)$, where σ^2 is known.

(a) Show that $\bar{X}$ is sufficient for θ.

(b) Is the class of p.d.f.'s $\{g(\bar{x};\theta), -\infty < \theta < \infty\}$ complete?

(c) Is $\bar{X}$ a unique best statistic for θ? Show why or why not.

4. Let X denote a sample of size 1 from $f(x;\theta) = \frac{\theta^x e^{-\theta}}{x!}$, $x = 0, 1, 2, \ldots$, $\theta > 0$. Define $\varphi(x) = 1$ if $x = 0$ and $\varphi(x) = 0$, otherwise.

(a) Find $E[\varphi(X)]$

(b) By using (a) find the unique best statistic for $e^{-\theta}$.

5. Suppose X is a sample of size 1 from the binomial p.d.f. $f(x;\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$, $x = 0, 1, \ldots, n$, $0 < \theta < 1$. Is there an unbiased estimator for $\frac{1}{\theta}$? (Hint: Set $E[u(X)] = \frac{1}{\theta}$.)

1. $Y \sim \frac{\sum^{n} (x-\mu)^2}{\sigma^2} \sim \chi^2_{n}$

(a) FIND a & b $\ni$ $P_r[a \le \chi^2_n \le b] = \alpha$

(OR $\int_{-\infty}^{a} \chi^2_n \, dY = \int_{b}^{\infty} \chi^2 \, dY = \frac{1-\alpha}{2}$)

$$\Rightarrow P_r\left[a \le \frac{\sum(x_i-\mu)^2}{\sigma^2} < b\right] = \alpha$$

$$= P_r\left[\frac{1}{b} \le \frac{\sigma^2}{\sum(x_i-\mu)^2} \le \frac{1}{a}\right]$$

$$= P_r\left[\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{b} \le \sigma^2 \le \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{a}\right]$$

$$CI = \left[\frac{\sum(x_i-\mu)^2}{b}, \frac{\sum(x_i-\mu)^2}{a}\right]$$

(b) $\frac{ns^2}{\sigma^2} \sim \chi^2_{n-1}$

again, find a & b $\ni$

$$\alpha = P_r[a \le \chi^2_{n-1} \le b]$$

$$\Rightarrow \alpha = P_r\left[a \le \frac{ns^2}{\sigma^2} \le b\right]$$

$$= P_r\left[\frac{1}{b} \le \frac{\sigma^2}{ns^2} \le \frac{1}{a}\right]$$

$$= P_r\left[\frac{ns^2}{b} \le \sigma^2 \le \frac{ns^2}{a}\right]$$

$$\Rightarrow CI = \left[\frac{ns^2}{b}, \frac{ns^2}{a}\right]$$

$$s^2 = \frac{\sum(x_i-\bar{x})^2}{n}$$

(c) $f(x) = \chi^2_{n-1}$

$$L = \frac{ns^2}{b} - \frac{ns^2}{a}$$

$$= ns^2\left(\frac{1}{b} - \frac{1}{a}\right)$$

Minimize L SUBJECT TO

$$\int_a^b f(x)\,dx - 0.95 = 0$$

Lagrange multiplier

$$L_A = ns^2\left(\frac{1}{b} - \frac{1}{a}\right) + \lambda\left[\int_a^b f(x)\,dx - 0.95\right]$$

$$\frac{\delta L_A}{\delta a} = -ns^2 \frac{d}{da}\left(\frac{1}{a}\right) + \lambda\left[-f(a)\right] = 0$$

$$= -ns^2\left(\frac{-1}{a^2}\right) - \lambda f(a) = 0$$

$$\frac{ns^2}{a^2} - \lambda f(a) = 0 \qquad (1)$$

$$\frac{\delta L_A}{\delta b} = ns^2 \frac{-1}{b^2} + \lambda f(b) = 0 \qquad (2)$$

$$\cancel{\frac{\delta L_A}{\delta \lambda} = \int_a^b f(x)\,dx - 0.95 = 0}$$

IN (1)

$$\lambda f(a) = \frac{ns^2}{a^2} \Rightarrow \lambda = \frac{ns^2}{f(a)a^2}$$

Substitute into (2):

$$-\frac{\cancel{ns^2}}{b^2} + \frac{\cancel{ns^2}\, f(b)}{f(a)\,a^2} = 0.$$

$$\Rightarrow +\frac{1}{b^2} = \frac{f(b)}{f(a)a^2} \Rightarrow f(a)\,a^2 = f(b)\,b^2$$

(d) IN (a)

$$L_a = \Sigma(x_i - \mu)^2\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$= \frac{\Sigma(x_i-\mu)^2}{\sigma^2} \sigma^2\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$= \underbrace{\frac{\Sigma(x_i-\mu)^2}{\sigma^2}}_{\sim \chi^2_n} \sigma^2\left(\frac{1}{a} - \frac{1}{b}\right)$$

$E[\chi^2_r] = r$

$$E[L_a] = \sigma^2\left(\frac{1}{a} - \frac{1}{b}\right)(n)$$ ✓

IN (b)

$$L_b = ns^2\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$= \underbrace{\frac{ns^2}{\sigma^2}}_{\sim\chi^2_{n-1}}\left[\frac{1}{a} - \frac{1}{b}\right]\sigma^2$$

$$E[L_b] = (n-1)\left[\frac{1}{a} - \frac{1}{b}\right]\sigma^2$$ ✓

$$\begin{cases} \operatorname{Var} L_a = \left[\sigma^2\left(\frac{1}{a} - \frac{1}{b}\right)\right]^2 \operatorname{Var} \chi^2_n = 2n\left[\sigma^2\left(\frac{1}{a} - \frac{1}{b}\right)\right]^2 \\ \operatorname{Var} L_b = \left[\sigma^2\left(\frac{1}{a} - \frac{1}{b}\right)\right]^2 \operatorname{Var} \chi^2_{n-1} = 2(n-1)\left[\sigma^2\left(\frac{1}{a} - \frac{1}{b}\right)\right]^2 \end{cases}$$

(What about Var ?)

μ $\bar{x}$

$$\left.\begin{array}{l} E\{L_a\} > E[L_b] \\ \operatorname{Var}\{L_a) > \operatorname{Var} L_b \end{array}\right\rangle$$ This doesn't make sense!

2. $X \sim n(\mu_1, \sigma_1^2)$ $\qquad$ $Y \sim n(\mu_2, \sigma_2^2)$

μ_1 KNOWN $\qquad$ μ_2 unknown

$$Z_1 = \frac{\Sigma(x_i - \mu_1)^2}{\sigma_1^2} \sim \chi^2_n \checkmark \qquad Z_2 \; \frac{mS^2}{\sigma_2^2} \sim \chi^2_{m-1} \checkmark$$

$$\frac{Z_1}{Z_2} = \frac{\sum^n (x_i - \bar{x})^2}{m\, S^2} \quad \frac{\sigma_2^2}{\sigma_1^2}$$

F STATISTIC $\longrightarrow$ $$\frac{Z_1/n}{Z_2/(m-1)} = \underbrace{\frac{\Sigma(x_i - \bar{x})^2}{m\, S^2}\left(\frac{m-1}{n}\right)}_{\text{CALL THIS THING } R} \frac{\sigma_2^2}{\sigma_1^2} \sim F_{n,m-1}$$

FIND $a \,\&\, b$

$\ni$ $Pr[a \le F_{n,m-1} \le b] = 0.95 = \alpha$

THEN

$$Pr\left[a \le R\frac{\sigma_2^2}{\sigma_1^2} \le b\right]$$

$$= Pr\left[\frac{a}{R} \le \frac{\sigma_2^2}{\sigma_1^2} \le \frac{b}{R}\right] = \alpha$$

ie

$$\int_0^a F_{n,m-1} = \int_b^\infty F_{n,m-1} = \frac{1-\alpha}{2}$$

3. $X_i \sim n(\theta, \sigma^2)$

(a) $X_i \sim \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$\prod_{i=1}^{n} f(x_i;\theta) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-\frac{\sum_{i=1}^{n}(x_i-\theta)^2}{2\sigma^2}}$$

$$= (\quad)^n e^{\frac{\sum^n [x_i^2 - 2\theta x_i + \theta^2]}{-2\sigma^2}}$$

$$= (\quad)^n e^{\frac{\sum^n x_i^2 - 2\theta \sum x_i + n\theta^2}{-2\sigma^2}}$$

$$= \underbrace{\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{\frac{-2\theta n\bar{x}}{-2\sigma^2}} e^{-\frac{n\theta^2}{2\sigma^2}}}_{K_1[U_1(x_1 \dots x_n;\theta)] > 0} \underbrace{e^{\frac{\sum^n x_i^2}{-2\sigma^2}}}_{K_2[U_2(x_1 \dots x_n)] > 0}$$

✓

$\therefore$ SUFFICIENT

(b) $g(\bar{x};\theta) = n(\theta, \sigma^2/n)$

$$E[U(\bar{x})] = \frac{1}{\sqrt{\frac{2\pi}{n}}\,\sigma} \int_{-\infty}^{\infty} U(x)\, e^{-\frac{(\bar{x}-\theta)^2}{2\sigma^2/n}} dx$$

$$x' = \bar{x} - \theta$$

✓ $$E[U(x)] = \frac{1}{\sqrt{\frac{2\pi}{n}}\,\sigma} \int_{-\infty}^{\infty} U(x+\theta)\, e^{-\frac{(x)^2}{2\sigma^2/n}} dx = 0$$

$\Rightarrow U \neq 0$ MUST DEPEND on θ for integral to vanish.
$\therefore U=0$ and class is complete

c. [$\bar{x}$ is sufficient and
√ $g(\bar{x};\theta)$ is complete.]

Since $E[\bar{x}] = \theta$ (unbiased)
$\bar{x}$ is the best statistic
for θ. (minimum variance
& unbiased)

since $\bar{x} \sim n(\underbrace{\theta}_{E[\bar{x}]}, \sigma^2/n)$

$$4.(a)\ X \sim f(x;\theta) = \frac{\theta^x e^{-\theta}}{x!} \quad ; x = 0, 1 \ldots \quad \theta > 0$$

$$\phi(x) = \begin{cases} 1 & ; x = 0 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$\phi(x) = \delta_{x,0} \quad (\text{KRONECKER } \delta)$$

$$(a)\ E[\phi(x)] = \sum_{x=0}^{\infty} \frac{\theta^x e^{-\theta}}{x!} \phi(x)$$

$$= \sum_{x=0}^{\infty} \frac{\theta^x e^{-\theta}}{x!} \delta_{x,0}$$

$$= \frac{\theta^0 e^{-\theta}}{0!} = e^{-\theta} \checkmark$$

(b) $Y_2 = \phi(x) \rightarrow$ UNBIASED FOR θ

~~Let Y_1 be sufficient~~

~~$E[Y_2 | Y_1] = \varphi(Y_1) \Rightarrow E[\varphi_1(Y_1)] = e^{-\theta}$~~

~~IS $Y_2 = \phi(x)$ sufficient?~~

~~$$f[Y_2] = \begin{cases} e^{-\theta} & ; Y_2 = 1 \\ 1 - e^{-\theta} & ; Y_2 = 0 \end{cases}$$~~

~~$$\prod_{i=1}^{n} f(x_i;\theta) = \frac{\theta^x e^{-\theta}}{x!} \quad ; x = 0,1,2,3 \ (ie, n = 1)$$~~

~~$$= \begin{cases} e^{-\theta} & ; x = 0 \\ \frac{\theta^x e^{-\theta}}{x!} & ; x = 1, 2, \ldots \end{cases}$$~~

(b) $\bar{X}$ is complete and sufficient statistic for θ.
Since $e^{-\theta}$ is a monotolonic mapping (1 to 1), $e^{-\bar{X}}$ is sufficient and complete statistic for $e^{-\theta}$. Now does $\exists\ \psi[e^{-\bar{x}}] \ni E[\psi] = e^{-\theta}$

(This is garbage)

-5

Let's try this:

$Y_2 = \phi(x) \to E[Y_2] = e^{-\theta}$

Let Y_1 be sufficient for $e^{-\theta}$

($Y_1 = e^{-\bar{X}}$ will do) $\Rightarrow \bar{X}_1 = -\ln Y_1$

$\frac{dx_1}{dY_1} = -\frac{1}{Y}$

$f(Y_1) = \left|\frac{1}{Y}\right| (n\theta)^{-\ln Y} e^{-n\theta} / (\ln Y)!$

$E[Y_2 / Y_1]$

ARG

$X_1 = \bar{X}$ suff. for θ

$f(x;\theta)$ is complete

$\Rightarrow \varphi(x)$ is unique.

IS $Y_2=\phi(X)$ SUFFICIENT FOR $e^{-\theta}$?

$\varphi(X)$ does not have to be suff. for θ

in order to be a unique best statistic.

$$f_Y(Y) = \begin{cases} e^{-\theta} & Y_2 = 1 \\ 1-e^{-\theta} & Y_2 = 0 \end{cases}$$

$$'' \quad = e^{-\theta}\delta_{Y_1} + (1-e^{-\theta})\delta_{Y_2}$$

$$\prod_{i=1}^{n} (\quad) = \frac{(\theta)^{\Sigma X} e^{-n\theta}}{\prod X_i!}$$

$\bar{X}$ suff

END TIME

5. $f(x;\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$; $x = 0, \dots, n$

Does $\exists$ a $\circ$ $U(x) \ni E[U(x)] = \frac{1}{\theta}$?

$$E[U(x)] = \frac{1}{\theta} = \sum_{x=0}^{n} \binom{n}{x}\theta^x(1-\theta)^{n-x} U(x)$$

OR $$\sum_{x=0}^{n} \binom{n}{x}\theta^{x+1}(1-\theta)^{n-x} U(x) = \underline{1}$$

Here, we have a (n+1'ST Polynomial in θ) = 1. ~~Let~~ All coefficients of θ^m ; $m \neq 0$, must be zero.

$$1 = \binom{n}{0}U(0)\,\theta(1-\theta)^n + \binom{n}{1}U(1)\,\theta^2(1-\theta)^{n-1} + \dots + \binom{n}{n}U(n)\,\theta^{n+1}$$

As can be seen $\exists$ NO θ^0 (ie ~~no~~ NO constant term). Every term in the polynomial has a θ^m ; $m \geq \underline{1}$. Thus the relation cannot be satisfied and $\exists$ no unbiased estimate of $\frac{1}{\theta}$.

Bob Marks

Math 5384 Take Home Exam due: March 15, 1977.

1. problem # 7.33, page 234

2. " # 7.39, " 237

3. " # 7.42, " 242

4. " # 7.43, " 243

5. " # 7.48, " 247

6. Let $f(x;\theta) = \dfrac{e^{-\theta}\theta^x}{x!}$, $\theta > 0$, $x = 0, 1, 2, \cdots$.

(a) Let $\lambda = \theta^2$. Determine whether $\bar{X}$ is sufficient for λ where $X_1, \ldots, X_n$ denotes a random sample from $f(x;\theta)$.

(b) Find $E(\bar{X})$.

(c) Find $E(\bar{X}^2)$.

(d) Use (b) and (c) to construct a unique best statistic for $\lambda = \theta^2$. Denote the resulting statistic by Y.

(f) Find σ_Y^2.

(g) Use the Cramér-Rao inequality to determine a lower bound for the variance of an unbiased estimator of $\lambda = \theta^2$.

(h) Is Y an efficient statistic for λ?
Use (f) and (g).

(i) Find the maximum likelihood statistic for λ.

7 Derive the Cramér-Rao lower bound analogous to (1) on page 248 of the text for the case where $Y = u(X_1, \dots, X_n)$ is not necessarily unbiased for θ. Denote the bias in Y by $b(\theta)$, i.e. $E(Y) = \theta + b(\theta)$.

8. An experimenter may test the potency of a certain drug by giving groups of animals injections of the drug at certain levels. Each animal is assumed to show either of two possible responses, positive or negative. The probability of a positive response to a dosage at level x is denoted by $P(x)$. Consider the case where the response curve is given by $P(x) = \dfrac{1}{1+e^{-(\alpha+\beta x)}}$, where $-\infty < \alpha < \infty$ and $\beta > 0$ are unknown parameters to be estimated. The experimenter chooses N dose levels, $x_1, x_2, \dots, x_N$, and assigns $n_1, n_2, \dots n_N$

animals, respectively, to those levels. Let $Y_1, Y_2, \ldots, Y_N$ denote the number of positive responses at respective levels $x_1, x_2, \ldots, x_N$. Then $Y_i \sim b(n_i, P(x_i))$, $i=1,2,\ldots,N$. Note that $x_1, \ldots, x_N$ and $n_1, \ldots, n_N$, are known quantities.

(a) Write the joint p.d.f. of $Y_1, \ldots, Y_N$.

(b) Find sufficient statistics for α and β.

(c) Discuss briefly how one would go about finding maximum likelihood estimates for α and β, and describe where the difficulty lies.

(d) Another criterion for determining estimates for α and β is the following:

Let $\hat{P}_i = \dfrac{Y_i}{n_i}$ be an estimate of P_i and determine α and β so as to minimize $\displaystyle\sum_{i=1}^{N} \frac{n_i\,(P(x_i) - \hat{P}(x_i))^2}{P(x_i)(1-P(x_i))}$

Discuss the ease of computation (if any) gained in using this approach.

(e) In an effort to reduce the computational demands in (c) and (d) a third criterion for obtaining estimates of α and β was proposed and is as follows:

Determine α and β so as to minimize

$$\sum_{i=1}^{N} n_i \hat{P}(x_i)(1-\hat{P}(x_i))\left[\ln\left\{\frac{P(x_i)}{1-P(x_i)}\right\} - \ln\left\{\frac{\hat{P}(x_i)}{1-\hat{P}(x_i)}\right\}\right]^2.$$

Find the estimates of α and β by using this approach.

(f) Let $N=3$, $n_1=n_2=n_3=10$, $x_1=-1$, $x_2=0$, $x_3=1$. Evaluate the estimates of α and β using (e) when $Y_1=0$, $Y_2=4$, $Y_3=9$. (Define $p(1-p)\ln\left(\frac{p}{1-p}\right)$ to be 0 if $p=0$ or 1).

(g) Are the estimates in (e) functions of the sufficient statistics?

(h) Discuss how one might improve on the maximum likelihood estimates by means of the Rao-Blackwell Theorem.

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BOB MARKS

MA · 5384

TAKE HOME EXAM

DUE: 3/15/77

1. (7-33) p. 234

IN 7-32, WE SHOWED THAT, IF

$$f(x;\theta) = e^{\theta K(x) + S(x) + q(\theta)} \quad ; a < x < b \ ; \gamma < \theta < \delta$$

THEN $M_Y(t) = e^{q(\theta) - q(\theta + t)} \quad ; \gamma < \theta + t < \delta$

WHERE $Y = K(X)$

GIVEN $E[Y] = \theta$

THUS $\frac{d}{dt} M_Y(t)\Big|_{t=0} = \theta = \frac{-dq(\theta+t)}{dt} e^{q(\theta) - q(\theta+t)}\Big|_{t=0}$

$= -dq(\theta+t)/dt \Big|_{t=0}$

SINCE $\frac{dq(\theta+t)}{dt} = \frac{dq(\theta+t)}{d\theta}$

THEN $\frac{dq(\theta+t)}{dt}\Big|_{t=0} = \frac{dq(\theta+t)}{d\theta}\Big|_{t=0} = \frac{dq(\theta)}{d\theta} = -\theta$

INTEGRATING: $q(\theta) = -\frac{1}{2}\theta^2 + C(t)$

WHERE $C(t)$ IS AN INTEGRATION CONSTANT, POSSIBLY A FUNCTION OF t. THUS

$$q(\theta + t) = -\frac{1}{2}(\theta + t)^2 + C(t)$$

SUBSTITUTING INTO MGF:

$$\begin{aligned} M_Y(t) &= e^{q(\theta) - q(\theta+t)} \\ &= e^{[-\frac{1}{2}\theta^2 + C(t)] - [-\frac{1}{2}(\theta+t)^2 + C(t)]} \\ &= e^{-\frac{1}{2}[\theta^2 - (\theta+t)^2]} \\ &= e^{+\frac{1}{2}[2\theta t + t^2]} = e^{\theta t + t^2/2} \end{aligned}$$

THE mgf FOR A NORMAL $[n(\mu, \sigma^2)]$ DISTRIBUTION IS $e^{\mu t + \sigma^2 t^2/2}$. THUS, WE CONCLUDE THAT $Y \sim n(\theta, 1)$.

2. (7-39) p. 237

$f(x_i;\theta) = \dfrac{\theta^{x_i} e^{-\theta}}{x_i!}$; $i = 1, 2, \ldots, n$

LET $Z = \sum_{i=2}^{n} X_i$

NOW, $M_X(t) = e^{\theta(e^t - 1)}$

$\Rightarrow M_Z(t) = e^{(n-1)\theta(e^t - 1)}$

THUS $f_1(z;\theta) = [\theta(n-1)]^z e^{-(n-1)\theta} / z!$

LET $Y = X_1 + Z$

SINCE X_i IS RANDOM SAMPLE, X_1 AND Z ARE INDEPENDENT. USING RESULTS FROM EXAMPLE 2 ON p. 124:

$Y_1 = Y = Z + X_1 \qquad Y_2 = X_1$

$\mu_1 = E[Z] = (n-1)\theta$

$\mu_2 = E[X_1] = \theta$

$$\Rightarrow g(Y, X_1) = \frac{[(n-1)\theta]^{Y - X_1} \theta^{X_1} e^{-n\theta}}{(Y - X_1)! \; X_1!}$$

$$= \frac{[(n-1)\theta]^Y}{(n-1)^{X_1}} \frac{e^{-n\theta}}{(Y - X_1)! \; X_1!}$$

NOW

$h(x_1 | Y) = g(y;x_1) / f(Y)$

WHERE

$$f_2(Y) = \frac{(n\theta)^Y e^{-n\theta}}{Y!} \quad ; Y = 0, 1, 2, \ldots$$

THUS

$$h(x_1|Y) = \frac{(n-1)^{Y-x_1} \cancel{\theta^Y} \cancel{e^{-n\theta}}}{x_1! \; (Y-x_1)!} \; \frac{Y!}{n^Y \cancel{\theta^Y} \cancel{e^{-n\theta}}}$$

$$= \frac{Y!}{x_1!\,(Y-x_1)!} \; \frac{(n-1)^{Y-x_1}}{n^Y}$$

NOTE THAT $h(x_1|Y)$ IS INDEPENDENT OF θ WHICH IT MUST, SINCE, AS SHOWN IN (7-7) p. 222, Y IS A SUFFICIENT STATISTIC FOR θ. AGAIN

$$h(x_1|Y) = \binom{Y}{x_1} \frac{(n-1)^{Y-x_1}}{n^Y} \quad ; \quad \begin{array}{l} x_1 = 0, 1, 2, \ldots, Y \\ Y = 0, 1, 2, \ldots \end{array}$$

LET
$$u(x_1) = \begin{cases} 1 & ; \quad x = 0, 1 \\ 0 & ; \quad \text{OTHERWISE} \end{cases}$$

THEN
$$E[u(x_1)] = \sum_{x_1=0}^{\infty} u(x_1) \frac{\theta^{x_1} e^{-\theta}}{x_1!}$$

$$= (1+\theta) e^{-\theta}$$

THEN

$$E[u(X_1) \mid Y=y] = \sum_{X_1} u(x_1)\, h(x_1 \mid y)$$

$$= u(0)\, h(0 \mid y) + u(1)\, h(1 \mid y)$$

$$= \binom{y}{0} \left(\frac{n-1}{n}\right)^{y} + \binom{y}{1} \frac{(n-1)^{y-1}}{n^{y}}$$

$$= \left(\frac{n-1}{n}\right)^{y} \left[1 + \frac{y}{(n-1)}\right]$$

IN ACCORDANCE WITH THEOREM 4 ON pg. 225, THE STATISTIC

$$\left(\frac{n-1}{n}\right)^{Y} \left[1 + \frac{Y}{n-1}\right]$$

IS THE BEST STATISTIC FOR $(1+\theta)e^{-\theta}$

3.(7.42) p. 242

$$(X_i, Y_i) = f(X, Y; \mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$$

$$f(X, Y; \mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}\left[\frac{1}{(1-\rho)^2}\left\{\left(\frac{X-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{X-\mu_1}{\sigma_1}\right)\left(\frac{Y-\mu_2}{\sigma_2}\right) + \left(\frac{Y-\mu_2}{\sigma_2}\right)^2\right]}$$

WE WILL NOW SHOW THAT f IS A MEMBER OF THE EXPONENTIAL CLASS:

$$f(X, Y; \mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$$

$$= \exp\left[\frac{-1}{2(1-\rho^2)}\left(\frac{X-\mu_1}{\sigma_1}\right)^2 + \frac{\rho}{1-\rho^2}\left(\frac{X-\mu_1}{\sigma_1}\right)\left(\frac{Y-\mu_2}{\sigma_2}\right) - \frac{1}{2(1-\rho^2)}\left(\frac{Y-\mu_2}{\sigma_2}\right)^2 - \ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right)\right]$$

$$= \exp\left[\frac{-1}{2\sigma_1^2(1-\rho^2)}\left(\underline{X}^2 - 2\underline{\mu_1}X + \underline{\mu_1}^2\right) + \frac{\rho}{\sigma_1\sigma_2(1-\rho^2)}\left\{\underline{XY} - \underline{\mu_2}X - \underline{\mu_1}Y + \mu_1\mu_2\right\} - \frac{1}{2\sigma_2^2(1-\rho^2)}\left(\underline{Y}^2 - 2Y\underline{\mu_2} + \underline{\mu_2}^2\right) - \ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right)\right)$$

$$= \exp\left[\left\{\frac{-1}{2\sigma_1^2(1-\rho^2)}\right\}X^2 + \left\{\frac{2\mu_1}{2\sigma_1^2(1-\rho^2)} - \frac{\mu_2\rho}{\sigma_1\sigma_2(1-\rho^2)}\right\}X\right.$$

$$+ \left\{\frac{\rho}{\sigma_1\sigma_2(1-\rho^2)}\right\}XY + \left\{\frac{1}{2\sigma_2^2(1-\rho^2)}\right\}Y^2 + \left\{\frac{2\mu_2}{2\sigma_2^2(1-\rho^2)} - \frac{\mu_1\rho}{\sigma_1\sigma_2(1-\rho^2)}\right\}Y$$

$$+ \left\{\frac{-\mu_1^2}{2\sigma_1^2(1-\rho^2)} + \frac{\rho\mu_1\mu_2}{\sigma_1\sigma_2(1-\rho^2)} - \frac{\mu_2^2}{2\sigma_2^2(1-\rho^2)}\right.$$

$$\left.\left. - \ln\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right)\right\}\right]$$

IN ACCORDANCE WITH THE NOTATION ON p. 242, WE HAVE

$$p_1(\mu_1, \mu_2 \sigma_1, \sigma_2 \rho) = \frac{-1}{2\sigma_1^2(1-\rho^2)} \quad ; K_1(X,Y) = X^2$$

$$p_2(\mu_1, \mu_2 \sigma_1, \sigma_2 \rho) = \left[\frac{\mu_1}{\sigma_1^2} - \frac{\mu_2 \rho}{\sigma_1 \sigma_2}\right] \frac{1}{1-\rho^2} \quad ; K_2(X,Y) = X$$

$$p_3(\mu_1, \mu_2 \sigma_1, \sigma_2 \rho) = \frac{\rho}{\sigma_1 \sigma_2 (1-\rho^2)} \quad ; K_3(X,Y) = XY$$

$$p_4(\mu_1, \mu_2 \sigma_1, \sigma_2 \rho) = \frac{-1}{2\sigma_2^2(1-\rho^2)} \quad ; K_4(X,Y) = Y^2$$

$$p_5(\mu_1, \mu_2 \sigma_1, \sigma_2 \rho) = \left[\frac{\mu_2}{\sigma_2^2} - \frac{\mu_1 \rho}{\sigma_1 \sigma_2}\right] \frac{1}{1-\rho^2} \quad ; K_5(X,Y) = Y$$

$$S(X,Y) = 0$$

$$q(\mu_1, \mu_2 \sigma_1, \sigma_2 \rho) = \frac{-\mu_1^2}{2\sigma_1^2(1-\rho^2)} + \frac{\rho \mu_1 \mu_2}{\sigma_1 \sigma_2 (1-\rho^2)} - \frac{\mu_2^2}{2\sigma_2^2(1-\rho^2)} - \ln\left(2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}\right)$$

THE SUFFICIENT STATISTICS ARE

$$Y_j = \sum_{j=1}^{N} K_j(x)$$

THUS

$$Y_1 = \sum^n X_i^2$$

$$Y_2 = \sum^n X_i$$

$$Y_3 = \sum^n X_i Y_i$$

$$Y_4 = \sum^n Y_i^2$$

$$Y_5 = \sum^n Y_i$$

NOW

$$\bar{X} = \frac{1}{n}\sum^n X_i = \frac{1}{n} Y_2$$

$$\bar{Y} = \frac{1}{n}\sum^n Y_i = \frac{1}{n} Y_5$$

BOTH OF THESE ARE ONE TO ONE MAPPINGS.

NOW:

$$S_1^2 = \frac{1}{n}\sum^n (X_i - \bar{X})^2$$

$$= \frac{1}{n}\left[\sum^n X_i^2 - 2\bar{X}\sum X_i + n\bar{X}^2\right]$$

$$= \frac{1}{n}\left[Y_1 - 2n\bar{X}^2 + n\bar{X}^2\right]$$

$$= \frac{1}{n} Y_1 - \bar{X}^2 = \frac{1}{n} Y_1 - \frac{1}{n^2} Y_2$$

FOR A GIVEN Y_1 AND Y_2, S_1^2 IS UNIQUE

SIMILARLY:

$$S_2^2 = \frac{1}{n} Y_4 - \bar{Y}^2 = \frac{1}{n} Y_4 - \frac{1}{n^2} Y_5^2$$

NOW:

$$\frac{1}{nS_1S_2}\sum^n (X_i - \bar{X})(Y_i - \bar{Y})$$

$$= \frac{1}{nS_1S_2}\sum^n\left[X_iY_i - \bar{X}Y_i - \bar{Y}X_i + \bar{X}\bar{Y}\right]$$

$$= \frac{1}{nS_1S_2}\left[\sum^n X_iY_i - \bar{X}\sum^n Y_i - \bar{Y}\sum^n X_i + n\bar{X}\bar{Y}\right]$$

$$= \frac{1}{nS_1S_2}\left[Y_3 - \left(\frac{1}{n}Y_2\right)Y_5 - \left(\frac{1}{n}Y_5\right)Y_2 + n\left(\frac{1}{n}Y_2\right)\left(\frac{1}{n}Y_5\right)\right]$$

$$= \frac{1}{nS_1S_2}\left[Y_3 - \frac{1}{n}Y_2Y_5\right] = $$

IN SUMMARY:

$$\bar{X} = \frac{1}{n} Y_2$$

$$\bar{Y} = \frac{1}{n} Y_5$$

$$S_1^2 = \frac{1}{n} Y_1 - \frac{1}{n^2} Y_2$$

$$S_2^2 = \frac{1}{n} Y_4 - \frac{1}{n^2} Y_5^2$$

$$\hat{\rho} = \frac{1}{nS_1S_2}\sum^n (X_i - \bar{X})(Y_i - \bar{Y}) = \frac{1}{nS_1S_2}\left[Y_3 - \frac{1}{n}Y_2Y_5\right]$$

THESE MAPPINGS ARE ONE TO ONE. THAT IS, FOR A GIVEN $Y_1, Y_2, Y_3, Y_4,$ & Y_5, $\exists$ UNIQUE $\bar{X}$, $\bar{Y}$, S_1^2, S_2^2 AND $\hat{\rho}$ AND VISA VERSA. THUS, $\bar{X}$, $\bar{Y}$, S_1^2, S_2^2 & $\hat{\rho}$ ARE ALSO JOINT SUFFICIENT STATISTICS FOR $\mu_1, \mu_2, \sigma_1, \sigma_2$, AND ρ.

4. (7·43) p. 243

$$f(x;\theta_1,\theta_2) = e^{p_1(\theta_1,\theta_2)K_1(x) + p_2(\theta_1,\theta_2)K_2(x) + S(x) + q(\theta_1,\theta_2)} \quad ; a < x < b$$

ASSUME $K_1'(x) = c K_2'(x)$

OR $\frac{d}{dx} K_1(x) = c \frac{d}{dx} K_2(x)$

INTEGRATING:

$$K_1(x) = c K_2(x) + c_2(\theta_1\theta_2)$$

WHERE c_2 IS AN INTEGRATION CONSTANT WHICH MIGHT POSSIBLY BE A FUNCTION (in general yes) OF θ_1 AND/OR θ_2. THUS

$$p_1(\theta_1,\theta_2)K_1(x) = c\, p_1(\theta_1,\theta_2)K_2(x) + c_3(\theta_1\theta_2)$$

WHERE $c_3(\theta_1\theta_2) = p_1(\theta_1\theta_2)\, c_2(\theta_1\theta_2)$

THEN

$$p_1(\theta_1\theta_2)\,K_1(x) + p_2(\theta_1,\theta_2)\,K_2(x)$$
$$= [c\,p_1(\theta_1\theta_2) + p_2(\theta_1\theta_2)]\,K_2(x) + c_3(\theta_1\theta_2)$$

LET

$$q_1(\theta_1,\theta_2) = c_3(\theta_1\theta_2) + q(\theta_1\theta_2)$$
$$p(\theta_1\theta_2) = c\,p_1(\theta_1\theta_2) + p_2(\theta_1\theta_2)$$

THEN OUR PDF CAN BE WRITTEN:

$$f(x;\theta_1\theta_2) = e^{p(\theta_1\theta_2)K_2(x) + S(x) + q_1(\theta_1\theta_2)} \quad ; a < x < b$$

5. (7-48) p. 247

$f(x;\theta) = \frac{1}{\theta} \quad ; 0 < x < \theta \quad ; 0 < \theta < \infty$

TO FIND THE JOINT PDF OF THE ORDER STATISTICS Y_1 AND Y_n, USE (3) ON p. 151 WITH $i = 1$ AND $j = n$

$$g(Y_1 Y_n) = \frac{n!}{(n-2)!} \left[F(Y_n) - F(Y_1)\right]^{n-2} f(Y_1) f(Y_n)$$

WHERE $\quad F(Y_n;\theta) = \frac{Y_n}{\theta} \quad ; 0 < x < \theta$

THUS

$$g(Y_1 Y_n) = \frac{n!}{(n-2)!} \left[\frac{Y_n}{\theta} - \frac{Y_1}{\theta}\right]^{n-2} \frac{1}{\theta^2} \quad ; 0 < Y_1 < Y_n < \theta$$

$$= \frac{n!}{(n-2)!} \frac{1}{\theta^n} \left[Y_n - Y_1\right]^{n-2}$$

LET $\quad Z = Y_1/Y_n$

THUS

$$M_Z(t) = E\left[e^{(\frac{Y_1}{Y_n})t}\right]$$

$$= \int_0^\theta \int_0^{Y_n} e^{\frac{Y_1}{Y_n}t} \left[\frac{n!}{(n-2)!} \frac{1}{\theta^n} (Y_n - Y_1)^{n-2}\right] dY_1\, dY_n$$

LET $\quad \hat{Y}_1 = Y_1/\theta \Rightarrow Y_1 = Y_n \Rightarrow \hat{Y}_1 = Y_n/\theta$

$$\Rightarrow M_Z(t) = \int_0^\theta \int_0^{Y_n/\theta} e^{\theta \hat{Y}_1 t/Y_n} \left[\frac{n!}{(n-2)!} \frac{1}{\theta^n} (Y_n - \hat{Y}_1\theta)^{n-2}\right] d(\theta \hat{Y}_1)\, dY_n$$

$$= \frac{n!}{(n-2)!} \frac{1}{\theta^{n-1}} \int_0^\theta \int_0^{Y_n/\theta} e^{\theta \hat{Y}_1 t/Y_n} (Y_n - \hat{Y}_1 \theta)^{n-2} d\hat{Y}_1\, dY_n$$

LET $\quad \hat{Y}_n = Y_n/\theta \Rightarrow Y_n = \theta \hat{Y}_n$

$Y_n = \theta \Rightarrow \hat{Y}_n = 1$

$$M_Z(t) = \frac{n!}{(n-2)!}\frac{1}{\Theta^{n-1}}\int_0^1\int_0^{\hat{Y}_n} e^{\frac{\Theta \hat{Y}_1 t}{\Theta \hat{Y}_n}} (\hat{Y}_n\Theta - \hat{Y}_1\Theta)^{n-2}\, d\hat{Y}_1\, d(\Theta\hat{Y}_n)$$

$$= \frac{n!}{(n-2)!}\frac{1}{\Theta^{n-2}}\int_0^1\int_0^{\hat{Y}_n} e^{\hat{Y}_1 t/\hat{Y}_n}\ \Theta^{n-2}(\hat{Y}_n - \hat{Y}_1)^{n-2}\, d\hat{Y}_1\, d\hat{Y}_n$$

$$= \frac{n!}{(n-2)!}\int_0^1\int_0^{\hat{Y}_n} e^{\frac{\hat{Y}_1}{\hat{Y}_n}t} (\hat{Y}_n - \hat{Y}_1)^{n-2}\, d\hat{Y}_1\, d\hat{Y}_n$$

$M_Z(t)$ IS INDEPENDENT OF Θ. THUS, $Z = Y_1/Y_n$ IS INDEPENDENT OF THE SUFFICIENT STATISTIC Y_n BY THEOREM 7 ON p. 234.

6. $f_1(x;\theta) = \dfrac{e^{-\theta}\theta^x}{x!}$; $\theta > 0$ $x = 0, 1, 2, \ldots$

(a) $\lambda = \theta^2 \Rightarrow \theta = \sqrt{\lambda}$

REDEFINE PDF AS

$$f(x;\lambda) = e^{-\sqrt{\lambda}}\lambda^{x/2}/x! \quad ; \lambda > 0 \quad , x = 0, 1, 2, \ldots$$

THEN

$$\prod_{i=1}^{n} f(x_i;\lambda) = e^{-n\sqrt{\lambda}}\lambda^{\frac{1}{2}\sum^n x_i} / \prod^n x_i!$$

$$= e^{-n\sqrt{\lambda}}\lambda^{\frac{n}{2}\bar{x}}\left[\prod^n x_i!\right]^{-1}$$

BY THEOREM 2 ON p. 219, $\bar{X}$ IS A SUFFICIENT STATISTIC FOR $\lambda = \theta^2$.* USING THE NOMENCLATURE THERE:

$$k_1[u_1(x_1, \ldots x_n);\lambda] = k_1[\bar{x};\lambda] = e^{-n\sqrt{\lambda}}\lambda^{\frac{n}{2}\bar{x}}$$

$$k_2[u_2(x_1, \ldots x_n)] = \left[\prod^n x_i!\right]^{-1}$$

(b) FOR $f_1(x;\theta)$, THE MGF IS

$$M_X(t;\theta) = e^{\theta(e^t - 1)}$$

$$\Rightarrow M_{\bar{X}}(t;\theta) = E\left[e^{t\frac{1}{n}\sum^n x_i}\right]$$

$$= E\left[\prod^n e^{tx_i/n}\right]$$

$$= \prod^n E\left[e^{tx_i/n}\right] \quad \text{(DUE TO IND.)}$$

$$= \left\{E\left[e^{tx_i/n}\right]\right\}^n$$

$$E\left[e^{tx_i/n}\right] = \sum_{i=0}^{\infty} e^{tx_i/n} f(x;\theta)$$

$$= e^{\theta[e^{t/n} - 1]}$$

$$\Rightarrow M_{\bar{X}}(t;\theta) = e^{n\theta[e^{t/n} - 1]}$$

OR

$$M_{\bar{X}}(t;\lambda) = e^{n\sqrt{\lambda}[e^{t/n} - 1]}$$

* THIS ALSO FOLLOWS FROM $f(x;\lambda)$ BEING IN THE EXP. CLASS WITH $K(x) = x/n$

(b) FINDING MOMENTS:

$$\frac{d}{d\bar{x}} M_{\bar{x}}(t) = \cancel{n}\theta \left[\frac{1}{\cancel{n}} e^{t/n}\right] e^{n\theta[e^{t/n}-1]}$$

$$= \theta e^{t/n} M_{\bar{x}}(t)$$

$$\therefore E[\bar{x}] = \frac{d}{d\bar{x}} M_{\bar{x}}(0) = \theta$$

(c) $$\frac{d^2}{d\bar{x}^2} M_{\bar{x}}(t) = \theta \left[\frac{1}{n} e^{t/n}\right] M_{\bar{x}}(t) + \theta e^{t/n} \frac{d}{dt} M_{\bar{x}}(t)$$

$$= \left[\frac{1}{n}\theta e^{t/n} + \theta^2 e^{2t/n}\right] M_{\bar{x}}(t)$$

$$\therefore E[\bar{x}^2] = \frac{d^2}{d\bar{x}^2} M_{\bar{x}}(0) = \frac{1}{n}\theta + \theta^2$$

(d) THE STATISTIC $Y = \bar{X}^2 - \frac{1}{n}\bar{X}$ IS UNBIASED:

$$E[Y] = E\left[\bar{X}^2 - \frac{1}{n}\bar{X}\right] = \theta^2$$

WE WILL NOW SHOW Y IS ALSO THE BEST STATISTIC FOR $\theta^2 = \lambda$.

(1) $f(x;\lambda)$ IS A MEMBER OF (DISCRETE) EXPONENTIAL CLASS:

$$f(x;\lambda) = e^{-\sqrt{\lambda}} \lambda^{x/2}/x! \quad ; x = 0, 1, 2, \ldots$$

$$= e^{\frac{x}{2}\ln\lambda - \ln x! - \sqrt{\lambda}}$$

$$\begin{cases} p(\lambda) = \frac{n}{2}\ln\lambda \\ k(x) = x/n \\ S(x) = -\ln x! \\ q(\lambda) = -\sqrt{\lambda} \end{cases}$$

(2) BY THEOREM 6 ON p. 232, $\sum^{n} K(x_i) = \bar{X}$ IS SUFF. STAT. FOR λ AND $g_1(\bar{x};\lambda)$ IS COMPLETE

(3) SINCE $g_1(\bar{x};\lambda)$ IS COMPLETE, $Y = \bar{X}^2 - \frac{1}{n}\bar{X}$ IS THE BEST STATISTIC FOR λ BY THEOREM 5, p. 229.

(f) $\sigma_Y^2 = E[Y^2] - E[Y]^2$

$E[Y] = \theta^2 \Rightarrow E[Y]^2 = \theta^4$

$E[Y^2] = E[(\bar{X}^2 - \frac{1}{n}\bar{X})^2]$

$= E[\bar{X}^4] - \frac{2}{n}E[\bar{X}^3] + \frac{1}{n^2}E[\bar{X}^2]$

FROM (c):

$\frac{d^2}{d\bar{x}^2}M_{\bar{x}}(t) = [\frac{1}{n}\theta e^{t/n} + \theta^2 e^{2t/n}]M_{\bar{x}}(t)$

THUS

$\frac{d^3}{d\bar{x}^3}M_{\bar{x}}(t) = [\frac{1}{n^2}\theta e^{t/n} + \frac{2\theta^2}{n}e^{2t/n}]M_{\bar{x}}(t)$

$+ [\frac{1}{n}\theta e^{t/n} + \theta^2 e^{2t/n}]\frac{d}{dt}M_{\bar{x}}(t)$

$= \{[\frac{1}{n^2}\theta e^{t/n} + \frac{2\theta^2}{n}e^{2t/n}]$

$+ [\frac{1}{n}\theta^2 e^{2t/n} + \theta^3 e^{3t/n}]\}M_{\bar{x}}(t)$

$= \{\theta^3 e^{3t/n} + \frac{3\theta^2}{n}e^{2t/n} + \frac{\theta}{n^2}e^{t/n}\}M_{\bar{x}}(t)$

$E[\bar{X}^3] = \frac{d^3}{d\bar{x}^3}M_{\bar{x}}(0) = \theta^3 + \frac{3\theta^2}{n} + \frac{\theta}{n^2}$

$\frac{d^4}{d\bar{x}^4}M_{\bar{x}}(t) = \{\frac{3}{n}\theta^3 e^{3t/n} + \frac{6\theta^2}{n^2}e^{2t/n} + \frac{\theta}{n^3}e^{t/n}\}M_{\bar{x}}(t)$

$+ \{\theta^3 e^{3t/n} + \frac{3\theta^2}{n}e^{2t/n} + \frac{\theta}{n^2}e^{t/n}\}\frac{d}{dt}M_{\bar{x}}(t)$

$= \{\frac{3\theta^3}{n}e^{3t/n} + \frac{6\theta^2}{n^2}e^{2t/n} + \frac{\theta}{n^3}e^{t/n}$

$+ \theta^4 e^{4t/n} + \frac{3\theta^3}{n}e^{3t/n} + \frac{\theta^2}{n^2}e^{2t/n}\}M_{\bar{x}}(t)$

$E[\bar{X}^4] = \frac{d^4}{d\bar{x}^4}M_{\bar{x}}(0)$

$= \theta^4 + \frac{6\theta^3}{n} + \frac{7\theta^2}{n^2} + \frac{\theta}{n^3}$

THUS

$E[Y^2] = [\theta^4 + \frac{6\theta^3}{n} + \frac{7\theta^2}{n^2} + \frac{\theta}{n^3}] - \frac{2}{n}[\theta^3 + \frac{3\theta^2}{n} + \frac{\theta}{n^2}]$

$+ \frac{1}{n^2}[\theta^2 + \frac{1}{n}\theta]$

$= \theta^4 + \theta^3[\frac{6}{n} - \frac{2}{n}] + \theta^2[\frac{7}{n^2} - \frac{6}{n^2} + \frac{1}{n^2}]$

$+ \theta[\frac{1}{n^3} - \frac{2}{n^3} + \frac{1}{n^3}]$

$= \theta^4 + \frac{4}{n}\theta^3 + \frac{2}{n^2}\theta^2$

$\therefore$

$\sigma_Y^2 = \frac{4}{n}\theta^3 + \frac{2}{n^2}\theta^2 = \frac{2}{n}\theta^2[2\theta + \frac{1}{n}]$

OR $\sigma_Y^2 = \frac{2}{n}\lambda[2\sqrt{\lambda} + \frac{1}{n}] = \frac{4\lambda^{3/2}}{n} + \frac{2\lambda}{n^2}$

(g) $f(x;\lambda) = \dfrac{e^{-\sqrt{\lambda}}\,\lambda^{x/2}}{x!}$

$\ln f(x;\lambda) = -\lambda^{\frac{1}{2}} + \frac{x}{2}\ln\lambda - \ln x!$

$\frac{\delta}{\delta\lambda}\ln f(x;\lambda) = -\frac{1}{2}\lambda^{-\frac{1}{2}} + \frac{x}{2\lambda}$

$\left[\frac{\delta}{\delta\lambda}\ln f(x;\lambda)\right]^2 = \frac{1}{4\lambda} - \cancel{2}\left(\frac{+1}{\cancel{2}\sqrt{\lambda}}\right)\left(\frac{x}{2\lambda}\right) + \frac{x^2}{4\lambda^2}$

$= \frac{1}{4\lambda} - \frac{x}{2\lambda^{3/2}} + x^2/4\lambda^2$

$E\left\{\left[\frac{\delta}{\delta\lambda}\ln f(x;\lambda)\right]^2\right\} = \frac{1}{4\lambda} - \frac{\sqrt{\lambda}}{2\lambda^{3/2}} + \frac{E[x^2]}{4\lambda^2}$

$E[x^2] = \text{var}\,x + E(x^2) = \theta + \theta^2$

$= \sqrt{\lambda} + \lambda$

$\therefore E\left\{\left[\frac{\delta}{\delta\lambda}\ln f(x;\lambda)\right\}^2\right] = \frac{1}{4\lambda} - \frac{1}{2\lambda} + \frac{\sqrt{\lambda}+\lambda}{4\lambda^2}$

$= -\frac{1}{4\lambda} + \frac{1}{4\lambda^{3/2}} + \frac{1}{4\lambda} = \frac{1}{4\lambda^{3/2}}$

$\sigma_{CR}^2 = \dfrac{1}{n\,E\left[\left\{\frac{\delta}{\delta\lambda}\ln f(x;\lambda)\right\}^2\right]} = \dfrac{4\lambda^{3/2}}{n}$

(h) ALTHOUGH Y IS THE BEST STATISTIC FOR λ, IT IS NOT EFFICIENT SINCE IT DOES NOT MEET THE CRAMÉR-RAO LOWER BOUND. THAT IS

$\sigma_{CR}^2 < \sigma_Y^2$

THE EFFICIENCY OF Y IS

$\eta = \dfrac{4\lambda^{3/2}}{4\lambda^{3/2} + \frac{2\lambda}{n^2}} = \dfrac{1}{1 + \frac{(2)\lambda}{n^2}\frac{1}{(4\lambda^{3/2}}}$

$= \dfrac{1}{1 + \frac{1}{2n^2\sqrt{\lambda}}}$

(i)

$$L(\lambda; x_1, x_2, \ldots x_n) = \frac{e^{-n\sqrt{\lambda}} \lambda^{\frac{1}{2}\Sigma x_i}}{\prod^n x_i!}$$

$$\ln L = -n\sqrt{\lambda} + \frac{n}{2}\bar{x} \ln \lambda - \ln \prod^n x_i!$$

$$\frac{\delta}{\delta\lambda} \ln L = \frac{-n}{2\lambda^{1/2}} + \frac{n\bar{x}}{2\lambda} = 0$$

$$\frac{n}{\lambda^{1/2}} = \frac{n\bar{x}}{\lambda} \Rightarrow \lambda^{\frac{1}{2}} = \bar{x}$$

OR

$$\hat{\lambda} = \bar{x}^2$$

THUS, THE MAXIMUM LIKELIHOOD ESTIMATE FOR λ IS $\hat{\lambda} = \bar{x}^2$

7. ① $1 = \int_{-\infty}^{\infty} f(x_i;\theta)\,dx_i \qquad i = 1, 2, \ldots, n$

$\theta + b(\theta) = \int_{-\infty}^{\infty} Y\, g(Y;\theta)\,dY = E[Y]$

$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u(x_1 x_2 \cdots x_n)\, f(x_1;\theta) \cdots f(x_n;\theta)\,dx_1 \cdots dx_n$

DIFFERENTIATE ① W.R.T. θ:

$0 = \int_{-\infty}^{\infty} \frac{\delta}{\delta\theta} f(x_i;\theta)\,dx_i$

$= \int_{-\infty}^{\infty} \left[\frac{\delta}{\delta\theta} \ln f(x_i;\theta)\right] f(x_i;\theta)\,dx_i$

② $= E\left[\frac{\delta}{\delta\theta} \ln f(x_i;\theta)\right]$

DIFF. ② WRT θ:

$1 + \frac{\delta b(\theta)}{\delta\theta} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u(x_1 \cdots x_n) \left[\sum_{i=1}^{n} \frac{1}{f(x_i;\theta)} \frac{\delta f(x_i;\theta)}{\delta\theta}\right]$

$\times f(x_1;\theta) \cdots f(x_n;\theta)\,dx_1 \cdots dx_n$

③ $= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u(x_1 \cdots x_n) \left[\sum_{i=1}^{n} \frac{\delta \ln f(x_i;\theta)}{\delta\theta}\right]$

$\times f(x_1;\theta) \cdots f(x_n;\theta)\,dx_1 \cdots dx_n$

④ LET $Z = \sum_{i=1}^{n} \frac{\delta}{\delta\theta} \ln f(x_i;\theta)$

FROM ②; $E[Z] = 0$

⑤ $\therefore \operatorname{Var} Z = n E\left[\left(\frac{\delta}{\delta\theta} \ln f(x;\theta)\right)^2\right] = \sigma_Z$

FROM ③; FOR $Y = u(x_1 \ldots x_n)$

$E[u(x_1 \ldots x_n) Z] = E[YZ] = 1 + \frac{\delta}{\delta\theta} b(\theta)$

NOW

$E[YZ] = 1 + \frac{\delta}{\delta\theta} b(\theta) = E(Y)\cancel{E(Z)}^{\,0} + \rho\, \sigma_Y \sigma_Z$

THUS

$\rho = \frac{1 + \frac{\delta}{\delta\theta} b(\theta)}{\sigma_Y \sigma_Z} \leq 1$

THUS

$\sigma_Y^2 \geq \frac{\left(1 + \frac{\delta b(\theta)}{\delta\theta}\right)^2}{\sigma_Z^2}$

OR, FROM ⑤

$$\sigma_Y^2 \geq \frac{\left[1 + \frac{\delta b(\theta)}{\delta\theta}\right]^2}{n E\left[\left\{\frac{\delta}{\delta\theta} \ln f(x;\theta)\right\}^2\right]}$$

8. X_i = LEVEL OF DOSAGE

$P(x_i) = P_r$[POSITIVE RESPONSE AT X_i]

$$= \frac{1}{1+e^{-(\alpha+\beta X_i)}} \quad ; -\infty < \alpha < \infty ,\ \beta > 0$$

n_i = # ANIMALS AT DOSAGE LEVEL X_i

Y_i = # ANIMALS WITH POS. RESPONSE AT X_i

$$Y_i \sim b(n_i ; P(x_i))$$

OR

$$f_i(Y_i ; \alpha, \beta) = \binom{n_i}{Y_i} [P(x_i)]^{Y_i} [1 - P(x_i)]^{n_i - Y_i}$$

$$= \binom{n_i}{Y_i} \frac{1}{[1+e^{-(\alpha+\beta X_i)}]^{Y_i}} \left[1 - \frac{1}{1+e^{-(\alpha+\beta X_i)}}\right]^{n_i - Y_i}$$

$$= \binom{n_i}{Y_i} \frac{1}{[1+e^{-(\alpha+\beta X_i)}]^{Y_i}} \left[\frac{e^{-(\alpha+\beta X_i)}}{1+e^{-(\alpha+\beta X_i)}}\right]^{n_i - Y_i}$$

$$= \binom{n_i}{Y_i} \frac{e^{Y_i(\alpha+\beta X_i)} [e^{-(\alpha+\beta X_i)}]^{n_i}}{[1+e^{-(\alpha+\beta X_i)}]^{n_i}}$$

$$= \binom{n_i}{Y_i} \frac{e^{Y_i(\alpha+\beta X_i)}}{[e^{\alpha+\beta X_i}+1]^{n_i}}$$

(a) ASSUMING Y_i'S ARE INDEPENDENT, WE HAVE

$$g(Y_1 \dots Y_N; \alpha, \beta) = \prod_{i=1}^{N} f_i(Y_i; \alpha, \beta)$$

$$= \prod_{i=1}^{N} \binom{n_i}{Y_i} \frac{e^{Y_i(\alpha + \beta X_i)}}{[e^{\alpha+\beta X_i} + 1]^{n_i}}$$

$$= e^{\alpha \Sigma Y_i + \beta \Sigma X_i Y_i} \prod_{i=1}^{N} \frac{\binom{n_i}{Y_i}}{[e^{\alpha+\beta X_i} + 1]^{n_i}}$$

(b) WE WILL LOOK AT THE SUFFICIENT STATISTICS FOR THE CASE OF ONE SAMPLE VECTOR $\vec{Y} = (Y_1, Y_2, \dots, Y_N)$. ALTHOUGH EACH Y_i IS NOT A SAMPLE FROM A FIXED DISTRIBUTION, THE FACT THAT EACH Y_i IS INDEPENDENT FROM EVERY OTHER $Y_j \neq Y_i$ LET'S US USE THE CASE OF SEVERAL PARAMETERS AS OUTLINED ON p. 237. THAT IS, WE CAN MATHEMATICALLY TREAT EACH Y_i AS A RANDOM SAMPLE OF SORTS. THUS, WE HAVE

$$\prod^{N} f_i(Y_i; \alpha, \beta) = g(\vec{Y}; \alpha, \beta)$$

$$= e^{\alpha \Sigma Y_i + \beta \Sigma X_i Y_i}$$

$$\times \prod_{i=1}^{N} \frac{\binom{n_i}{Y_i}}{[e^{\alpha+\beta X_i} + 1]^{n_i}}$$

IN THE DEVELOPMENT OF CRITERION FOR A STATISTIC'S SUFFICIENCY ON p. 238, THE ONLY DIFFERENCE BETWEEN OUR TREATMENT AND THE "RANDOM SAMPLE" TREATMENT IS THAT WE HAVE $f_i(Y_i;\alpha,\beta)$ INSTEAD OF $f(Y_i;\alpha,\beta)$. AS SUCH, WE MAY REWORD THE SUFFIENCY CONDITION AS FOLLOWS:

THE STATISTICS $Z_1 = U_1(\vec{Y})$ AND $Z_2 = U_2(\vec{Y})$ ARE JOINT SUFFICIENT STATISTICS FOR α AND β IFF $\exists$ TWO NON-NEGATIVE FUNCTIONS, k_1 & k_2 SUCH THAT

$$\prod_{i=1}^{N} f_i(x_i;\alpha,\beta) = k_1[U_1(\vec{Y}), U_2(\vec{Y}); \alpha,\beta]\, k_2[\vec{Y}]$$

THUS, LET

$$Z_1 = U_1(Y_1 \ldots Y_N) = \sum^N Y_i$$
$$Z_2 = U_2(Y_1 \ldots Y_N) = \sum^N x_i Y_i$$
$$k_2(\vec{Y}) = \prod^N \binom{n_i}{Y_i}\left[e^{\alpha+\beta x_i}+1\right]^{-n_i}$$

THEN

$$\prod^N f_i(x_i;\alpha,\beta) = e^{\alpha Z_1 + \beta Z_2}\, k_2(Y_1 \ldots Y_N)$$

THUS, Z_1 AND Z_2 ARE JOINT SUFFICIENT STATISTICS FOR α AND β.

(c) WE AGAIN CONSIDER THE CASE FOR $n=1$.

THE CORRESPONDING LIKELIHOOD FUNCTION IS:

$$L(Y_1, Y_2, \ldots, Y_N; \alpha, \beta) = e^{\alpha \sum^N Y_i + \beta \sum^N X_i Y_i} \cdot \prod^N \binom{n_i}{Y_i} \left[e^{\alpha + \beta X_i} + 1 \right]^{-n_i}$$

$$\ln L = \alpha \sum^N Y_i + \beta \sum^N X_i Y_i + \sum^N \ln \binom{n_i}{Y_i} - \sum^N n_i \ln\left[1 + e^{\alpha + \beta X_i}\right]$$

$$\frac{\delta \ln L}{\delta \alpha} = \sum^N \left[Y_i - \frac{n_i e^{\alpha + \beta X_i}}{1 + e^{\alpha + \beta X_i}} \right]$$

$$= \sum^N \left[Y_i - \frac{n_i}{e^{-(\alpha + \beta X_i)} + 1} \right] = 0$$

$$① \Rightarrow \sum^N Y_i = \sum^N \frac{n_i}{e^{-(\alpha + \beta X_i)} + 1}$$

$$\frac{\delta \ln L}{\delta \beta} = \sum^N \left[X_i Y_i - \frac{n_i X_i e^{\alpha + \beta X_i}}{1 + e^{\alpha + \beta X_i}} \right]$$

$$= \sum^N X_i \left[Y_i - \frac{n_i}{e^{-(\alpha + \beta X_i)} + 1} \right]$$

$$② \Rightarrow \sum^N X_i Y_i = \sum^N \frac{X_i n_i}{e^{-(\alpha + \beta X_i)} + 1}$$

TO DETERMINE THE MLE OF α & β, FOR A GIVEN SET OF X_i'S AND n_i'S, WE MUST SIMULTANEOUSLY SOLVE ① AND ② FOR α AND β. IN THE GENERAL CASE, THIS IS NOT POSSIBLE. EVEN FOR GIVEN X_i'S AND n_i'S, THE RESULT CALLS FOR SOLUTION OF TWO SIMULTANEOUS TRANCENDENTAL EQUATIONS. VERY MESSY. (FOR $n > 1$, IT WOULD BE UGLIER)

(d) $\hat{P}_i = Y_i / n_i$

WE WISH TO MINIMIZE

$$S = \sum_{i=1}^{N} \frac{n_i [P(x_i) - \hat{P}_i]^2}{P(x_i)[1 - P(x_i)]}$$

$$= \sum_{i=1}^{N} \frac{n_i \left[\frac{1}{1+e^{-(\alpha+\beta x_i)}} - \hat{P}_i\right]^2}{\frac{1}{1+e^{-(\alpha+\beta x_i)}}\left[\frac{e^{-(\alpha+\beta x_i)}}{1+e^{-(\alpha+\beta x_i)}}\right]}$$

$$= \sum_{i=1}^{N} \frac{n_i \left[\frac{1}{1+e^{-(\alpha+\beta x)}} - \hat{P}_i\right]^2 \left[1+e^{-(\alpha+\beta x_i)}\right]^2}{e^{-\alpha+\beta x_i}}$$

$$= \sum_{i=1}^{N} n_i \left[\left\{\frac{1}{1+e^{-(\alpha+\beta x)}} - \hat{P}_i\right\}\left\{1+e^{-(\alpha+\beta x_i)}\right\} e^{\frac{1}{2}(\alpha+\beta x)}\right]^2$$

$$= \sum_{i=1}^{N} n_i \left[\left(1 - \hat{P}_i\left\{1+e^{-(\alpha+\beta x_i)}\right\}\right) e^{\frac{1}{2}(\alpha+\beta x_i)}\right]^2$$

$$= \sum_{i=1}^{N} n_i \left[(1-\hat{P}_i) e^{\frac{1}{2}(\alpha+\beta x_i)} - \hat{P}_i e^{-\frac{1}{2}(\alpha+\beta x_i)}\right]^2$$

$$\frac{\delta S}{\delta \alpha} = \sum_{i=1}^{N} 2 n_i \left[(1-\hat{P}_i) e^{\frac{1}{2}(\alpha+\beta x_i)} - P_i e^{-\frac{1}{2}(\alpha+\beta x_i)}\right] \times \left[\frac{1}{2}(1-\hat{P}_i) e^{\frac{1}{2}(\alpha+\beta x_i)} + \frac{1}{2}\hat{P}_i e^{-\frac{1}{2}(\alpha+\beta x_i)}\right]$$

$$= \sum_{i=1}^{N} n_i \left[(1-\hat{P}_i)^2 e^{\alpha+\beta x} - \hat{P}_i^2 e^{-(\alpha+\beta x_i)}\right]$$

$$\frac{\delta S}{\delta \beta} = \sum^{N} 2 n_i \left[(1-\hat{P}_i) e^{\frac{1}{2}(\alpha+\beta x_i)} - P_i e^{-\frac{1}{2}(\alpha+\beta x_i)}\right] \left[\frac{1}{2} x_i (1-\hat{P}_i) e^{\frac{1}{2}(\alpha+\beta x_i)} + \frac{1}{2} x_i \hat{P}_i e^{-\frac{1}{2}(\alpha+\beta x_i)}\right]$$

$$= \sum^{N} n_i x_i \left[(1-\hat{P}_i)^2 e^{\alpha+\beta x_i} - \hat{P}_i^2 e^{-(\alpha+\beta x_i)}\right]$$

HERE, WE HAVE THE SAME PROBLEM: SOLUTION OF TWO TRANCENDENTAL EQUATIONS [FROM $\frac{\delta S}{\delta \alpha} = \frac{\delta S}{\delta \beta} = 0$] FOR α & β.

AGAIN, VERY MESSY.

(e) LET

$$S = \sum_{i=1}^{N} n_i \hat{P}_i (1-\hat{P}_i) \left[\ln \frac{P(X_i)}{1-P(X_i)} - \ln \frac{\hat{P}_i}{1-\hat{P}_i} \right]^2$$

NOW

$$\frac{P(X_i)}{1-P(X_i)} = \frac{\frac{1}{1+e^{-(\alpha+\beta X_i)}}}{1 - \frac{1}{1+e^{-(\alpha+\beta X_i)}}}$$

$$= \frac{1}{[1+e^{-(\alpha+\beta X_i)}] - 1} = e^{\alpha+\beta X_i}$$

THUS

$$\ln \frac{P(X_i)}{1-P(X_i)} = (\alpha + \beta X_i)$$

AND

$$S = \sum_{i=1}^{N} n_i \hat{P}_i (1-\hat{P}_i) \left[(\alpha+\beta X_i) - \ln \frac{\hat{P}_i}{1-\hat{P}_i} \right]^2$$

$$\left[\begin{array}{l} \dfrac{\delta S}{\delta \alpha} = \sum^{N} 2 n_i \hat{P}_i (1-\hat{P}_i) \left[(\alpha+\beta X_i) - \ln \dfrac{\hat{P}_i}{1-\hat{P}_i} \right] \\ \dfrac{\delta S}{\delta \beta} = \sum^{N} 2 X_i n_i \hat{P}_i (1-\hat{P}_i) \left[(\alpha+\beta X_i) - \ln \dfrac{\hat{P}_i}{1-\hat{P}_i} \right] \end{array} \right.$$

FOR $\frac{\delta S}{\delta \alpha} = 0$:

$$① \quad \alpha \sum^{N} 2 n_i \hat{P}_i (1-\hat{P}_i) = -\beta \sum 2 X_i n_i \hat{P}_i (1-\hat{P}_i) + \sum^{N} 2 n_i \hat{P}_i (1-\hat{P}_i) \ln \frac{\hat{P}_i}{1-\hat{P}_i}$$

FOR $\frac{\delta S}{\delta \beta} = 0$:

$$② \quad \alpha \sum^{N} 2 X_i n_i \hat{P}_i (1-\hat{P}_i) = -\beta \sum^{N} 2 X_i^2 n_i \hat{P}_i (1-\hat{P}_i) + \sum^{N} 2 X_i n_i \hat{P}_i (1-\hat{P}_i) \ln \frac{\hat{P}_i}{1-\hat{P}_i}$$

LET

$$a_0 = \sum^{N} n_i \hat{P}_i (1 - \hat{P}_i)$$
$$a_1 = \sum^{N} X_i\, n_i \hat{P}_i (1 - \hat{P}_i)$$
$$a_2 = \sum^{N} X_i^2\, n_i \hat{P}_i (1 - \hat{P}_i)$$
$$b_0 = \sum^{N} n_i \hat{P}_i (1 - \hat{P}_i) \ln \hat{P}_i / 1 - \hat{P}_i$$
$$b_1 = \sum^{N} X_i\, n_i \hat{P}_i (1 - \hat{P}_i) \ln \hat{P}_i / 1 - \hat{P}_i$$

THEN FROM 1 AND 2:

$$\begin{bmatrix} a_0 \alpha = -a_1 \beta + b_0 \\ a_1 \alpha = -a_2 \beta + b_1 \end{bmatrix}$$

OR

$$\frac{a_0}{a_1} \alpha = -\beta + \frac{1}{a_1} b_0$$
$$a_1/a_2 \alpha = -\beta + \frac{1}{a_2} b_1$$

$$\Rightarrow \left(\frac{a_0}{a_1} - \frac{a_1}{a_2}\right)\alpha = \left(\frac{b_0}{a_1} - \frac{b_1}{a_2}\right)$$
$$(a_0 a_2 - a_1^2)\alpha = b_0 a_2 - b_1 a_1$$
$$\therefore\ \hat{\alpha} = \frac{b_0 a_2 - b_1 a_1}{a_0 a_2 - a_1^2}$$

$$\hat{\beta} = \frac{b_1}{a_2} - \frac{a_1}{a_2} \hat{\alpha}$$
$$= \frac{b_1}{a_2} - \frac{a_1}{a_2} \left[\frac{b_0 a_2 - b_1 a_1}{a_0 a_2 - a_1^2}\right]$$

(f) $N = 3$, $n_i = 10$ $i = 1, 2, 3$

$X_1 = -1$ $\quad X_2 = 0$ $\quad X_3 = 1$

$Y_1 = 0$ $\quad Y_2 = 4$ $\quad Y_3 = 9$

NOW $\ln \frac{\hat{p}_i}{1-\hat{p}_i} = \ln \frac{Y_i/n_i}{1 - Y_i/n_i} = \ln \frac{Y_i}{n_i - Y_i}$

COMPUTING NUMBERS:

$$a_0 = \sum^3 n_i \hat{p}_i (1 - \hat{p}_i)$$
$$= 10 \sum^3 \hat{p}_i (1 - \hat{p}_i)$$
$$= 10 \sum^3 \frac{Y_i}{n_i}\left(1 - \frac{Y_i}{n_i}\right)$$
$$= \sum^3 Y_i (1 - Y_i/10)$$
$$= \frac{1}{10} \sum^3 Y_i (10 - Y_i)$$
$$= \frac{1}{10} [0(10-0) + 4(10-4) + 9(10-9)]$$
$$= \frac{1}{10} [24 + 9] = \frac{33}{10} = 3.3$$

$$a_1 = \frac{1}{10} \sum^3 X_i Y_i (10 - Y_i)$$
$$= \frac{1}{10} [(-1)(0)(10) + (0)(4)(6) + (1)(9)(1)]$$
$$= \frac{9}{10} = 0.9$$

$$a_2 = \frac{1}{10} \sum^3 X_i^2 Y_i (10 - Y_i)$$
$$= \frac{1}{10} [(1)(0)(10) + (0)(4)(6) + (1)(9)(1)]$$
$$= 9/10 = 0.9$$

$$b_0 = 10 \sum^3 \hat{p}_i (1 - \hat{p}_i) \ln \frac{Y_i}{n_i - Y_i}$$
$$= 10 \sum^3 \frac{Y_i}{10}\left(1 - \frac{Y_i}{10}\right) \ln \frac{Y_i}{n_i - Y_i}$$
$$= \frac{1}{10} \sum^3 Y_i (10 - Y_i) \ln \frac{Y_i}{n_i - Y_i}$$
$$= \frac{1}{10} \left[(0)(10) \ln \frac{0}{10} + (4)(6) \ln \frac{4}{6} + (9)(1) \ln \frac{9}{1}\right]$$
$$= 1.004386$$

$$b_1 = \frac{1}{10} \sum^3 X_i Y_i (10 - Y_i) \ln \frac{Y_i}{10 - Y_i}$$
$$= \frac{1}{10}\left[(-1)(0)(10) \ln 0 + (0)(4)(6) \ln \frac{4}{6} + (1)(9)(1) \ln \frac{9}{1}\right]$$
$$= \frac{1}{10} \; 9 \times \ln 9$$
$$= 1.977502$$

$$\hat{\alpha} = \frac{b_0 a_2 - b_1 a_1}{a_0 a_2 - a_1^2}$$

$$= \frac{(1.004386)(0.9) - (1.977502)(0.9)}{(3.3)(0.9) - (0.9)^2}$$

$$= -0.405465$$

$$\hat{\beta} = \frac{b_1}{a_2} - \frac{a_1}{a_2}\hat{\alpha}$$

$$= \frac{1.977502}{0.9} + \frac{0.9}{0.9}(0.405465)$$

$$= 2.60269$$

(g) THE ESTIMATOR'S IN (e) ARE <u>NOT</u> A FUNCTION OF THE SUFFICIENT STATISTICS ΣY_i AND $\Sigma X_i Y_i$.

(h) WE COULD IMPROVE THE ESTIMATES OF α AND β BY APPLICATION OF THE RAO-BLACKWELL THEOREM FORM AS STATED ON p. 225 IN THEOREM 4. THAT IS, IF $\hat{\alpha}$ AND $\hat{\beta}$ ARE UNBIASED ESTIMATORS OF α AND β, THEN WE DEFINE THE NEW ESTIMATORS $\hat{\hat{\alpha}}$ AND $\hat{\hat{\beta}}$ SUCH THAT:

$$\hat{\hat{\alpha}} = E[\hat{\alpha} \mid Z_1, Z_2]$$

$$\hat{\hat{\beta}} = E[\hat{\beta} \mid Z_1, Z_2]$$

WHERE $Z_1 = \sum^{N} Y_i$ AND $Z_2 = \sum^{N} X_i Y_i$ ARE THE JOINT SUFFICIENT STATISTICS FOR α AND β. FROM THEM. 4 ON p. 225, $\hat{\hat{\alpha}}$ AND $\hat{\hat{\beta}}$ WILL BE UNBIASED ESTIMATORS OF α & β WITH SMALLER VARIANCES THAN $\hat{\alpha}$ AND $\hat{\beta}$.

Final Exam 5/3/77 Name: Bob MARKS

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1. Let $X_1, X_2, \ldots, X_n$ denote a random sample from $f(x) = \lambda e^{-\lambda x}$, $x > 0$. Suppose the only information available to the experimenter is $Y_1 = \min(X_1, \ldots, X_n)$. (a) Show that $\left(\frac{a}{nY_1}, \frac{b}{nY_1}\right)$ is a confidence interval for λ with confidence coefficient $e^{-a} - e^{-b}$, $(0 \le a < b)$. (b) If $e^{-a} - e^{-b} = 1 - \alpha$ is fixed, how should a and b be chosen to minimize the width of the confidence interval?

(a) $g(Y_1) = n f(Y_1)[1 - F(Y_1)]^{n-1}$ $= n\lambda e^{-\lambda x}[e^{-\lambda x}]^{n-1} = n\lambda e^{-n\lambda x}$

$F(Y_1) = 1 - e^{-\lambda x} \Rightarrow g(Y_1) = n\lambda e^{-n\lambda x}$

$f(z) = e^{-z}$

$x = \frac{z}{n\lambda} \Rightarrow z = n\lambda x \quad \frac{dz}{dx} = n\lambda$

NOW $z = n\lambda Y_1$

$Y_1 = \frac{z}{n\lambda} \Rightarrow \frac{dY_1}{dz} = \frac{1}{n\lambda}$

$g_z(z) = e^{-z}$

$z = n\lambda Y_1 \sim e^{-z}$

$\alpha = P_r[a < z < b] = \int_a^b e^{-z}\,dz = -e^{-z}\Big|_a^b = e^{-a} - e^{-b}$

$= P_r[a < n\lambda Y_1 < b] = P_r\left[\frac{a}{nY_1} < \lambda < \frac{b}{nY_1}\right]$

(b) $W = b - a$ SUBJECT TO $e^{-a} - e^{-b} - 1 + \alpha = 0$

$e^{-a} = e^{-b} + (1-\alpha)$

$a = -\ln[e^{-b} + (1-\alpha)]$

GARBAGE

$W = b + \ln[e^{-b} + (1-\alpha)]$

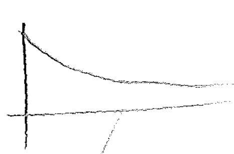
$\frac{dW}{db} = 1 + \frac{-e^{-b}}{e^{-b} + (1-\alpha)} = 0 \Rightarrow \frac{e^{-b}}{e^{-b} + (1-\alpha)} = 1$

$e^{-b} = e^{-b} + (1-\alpha) = 0$

$W = (b-a) + \lambda[e^{-a} - e^{-b} - 1 + \alpha]$

$\frac{dW}{da} = -1 - \lambda e^{-a} = 0 \Rightarrow 1 = -\lambda e^{-a} \Rightarrow \lambda = -e^{a}$

$\frac{dW}{db} = 1 - \lambda e^{-b}$



By inspection, choose $a = 0$

$e^{-a} - e^{-b} = 1 - \alpha$

$\Rightarrow e^{-b} = e^{-a} - (1-\alpha)$

$b = -\ln[e^{-a} - (1-\alpha)] = -\ln[\alpha] \Rightarrow$ $a = 0$, $b = \ln\frac{1}{\alpha}$

1. Let $X_1, X_2, \ldots, X_n$ denote a random sample from $f(x;\theta) = \frac{x}{\theta^2} e^{\ldots}$, $x > 0$, $\theta > 0$. (a) Find a maximum likelihood statistic for θ. (b) Find a sufficient statistic Y for θ. (c) Find a function of Y, $k(Y)$, which is unbiased for θ^2. (d) Find $Var(k(Y))$. (e) Find the Cramer-Rao lower bound for $Var(k(Y))$. (f) Is $k(Y)$ efficient for θ^2? (g) Is $k(Y)$ the unique best statistic for θ^2? (h) Determine the best critical region for testing $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1 > \theta_0$.

(a) $L(x_1 \ldots x_n;\theta) = \prod^n \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} = \frac{\prod x_i}{\theta^{2n}} e^{-\sum x^2/2\theta^2}$

$\ln L = \ln \prod^n x_i - 2n \ln\theta - \sum^n x^2/2\theta^2$

$\frac{\partial}{\partial\theta} \ln L = -\frac{2n}{\theta} - \frac{1}{2}\sum^n x^2 \frac{d}{d\theta}\theta^{-2} = -\frac{2n}{\theta} - \frac{1}{2}\sum^n x_i^2 (-2)\theta^{-3}$

$= -\frac{2n}{\theta} + \frac{\sum^n x_i^2}{\theta^3} = 0 \Rightarrow \frac{2n}{\theta} = \frac{\sum^n x_i^2}{\theta^3} \Rightarrow 2n\theta^2 = \sum^n x_i^2 \Rightarrow \hat{\theta} = +\sqrt{\frac{\sum^n x_i^2}{2n}}$

(b) $L(x_1 \ldots x_n;\theta) = \frac{\prod^n x_i}{\theta^{2n}} e^{-\frac{\sum^n x_i^2}{2\theta^2}}$

$= \underbrace{\prod^n x_i}_{K_2[x_1 \ldots x_n]} \underbrace{\left[\frac{1}{\theta^{2n}} e^{-\frac{\sum^n x_i^2}{2\theta^2}}\right]}_{k_1[u_1(x_1 \ldots x_n);\theta]}$

$\therefore Y = u_1(x_1 \ldots x_n) = \sum^n x_i^2$

$E(X) = \int_0^\infty \frac{x^2}{\theta^2} e^{-\frac{x^2}{2\theta^2}}$

$Var\, X^2 = 2\theta^2 - \frac{\pi}{2}\theta^2 = (2 - \frac{\pi}{2})\theta^2$

Thus from p. 166 corollary

$Var\, Y = n(2 - \frac{\pi}{2})\theta^2$

$= E[($

(c) $E[x_i^2] = \int_0^\infty \frac{x^3}{\theta^2} e^{-x^2/2\theta^2} dx$

$u = x^2 \quad dv = \frac{x}{\theta^2} e^{-x^2/2\theta^2} dx$

$du = 2x\,dx \quad v = -e^{-x^2/2\theta^2}$

$E[x^2] = -x^2 e^{-x^2/2\theta^2}\Big|_0^\infty + \int_0^\infty 2x e^{-x^2/2\theta^2} dx$

$= 2\theta^2 e^{-x^2/2\theta^2}\Big|_0^\infty$

$\int_0^\infty x^3 e^{-\frac{x^2}{2\theta^2}} dx = 2\theta^4$

$= 2\theta^2$ → OVER

OVER →

$E[X] = \int_0^\infty \frac{x^2}{\theta^2} e^{-x^2/2\theta^2} dx$

$u = x \quad dv = \frac{x}{\theta^2} e^{-x^2/2\theta^2}$

$du = dx \quad v = -e^{-x^2/2\theta^2}$

$E(X) = -x e^{-x^2/2\theta^2}\Big|_0^\infty + \int_0^\infty e^{\frac{x^2}{2\theta^2}}$

$= \frac{1}{2}\int_{-\infty}^\infty e^{\frac{x^2}{2\theta^2}} = \frac{1}{2}\sqrt{2\pi}\,\theta \underbrace{\frac{1}{\sqrt{2\pi}\theta}\int e^{\frac{-x^2}{2\theta^2}}}_{1} = \sqrt{\frac{\pi}{2}}\,\theta$

$$\frac{1}{nE\left[\left\{\frac{\delta}{\delta\theta}\ln f(x;\theta)\right\}^2\right]}$$

c

$$\ln f(x;\theta) = \ln \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} = \ln x - 2\ln\theta - \frac{x^2}{2\theta^2}$$

$$\frac{\delta}{\delta\theta}\ln f(x;\theta) = -\frac{2}{\theta} - \frac{x^2}{2}\frac{d}{d\theta}\theta^{-2} = -\frac{2}{\theta} - \frac{x^2}{2}(-2)\theta^{-3} = -\frac{2}{\theta} + \frac{x^2}{\theta^3}$$

$$\left[\frac{\delta}{\delta\theta}\ln f(x;\theta)\right]^2 = \left(\frac{x^2}{\theta^3} - \frac{2}{\theta}\right)^2 = \frac{x^4}{\theta^6} - \frac{4x^2}{\theta^4} + \frac{4}{\theta^2}$$

$$E\left[\left\{\frac{\delta}{\delta\theta}\ln f(x;\theta)\right\}^2\right] = \frac{1}{\theta^6}E[x^4] - \frac{4}{\theta^4}E[x^2] + \frac{4}{\theta^2}$$

$k(Y)$ unbiased for θ^2.

(e)

$$f(x;\theta) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}$$

LET $\hat{\theta} = \theta^2$

$$f(x;\hat{\theta}) = \frac{x}{\theta} e^{-\frac{x^2}{2\theta}}$$

$$\ln f(x;\hat{\theta}) = \ln x - \ln\theta - \frac{x^2}{2}\theta^{-1}$$

$$\left(\frac{\delta}{\delta\theta}\ln f(x;\hat{\theta})\right)^2 = \left(\frac{-1}{\theta} + \frac{x^2}{2\theta^2}\right)^2 = \frac{x^4}{4\theta^4} - \frac{x^2}{\theta^3} + \frac{1}{\theta^2}$$

$$E[(\quad)^2] = \frac{1}{4\theta^4}E[x^4] - \frac{1}{\theta^3}E[x^2] + \frac{1}{\theta^2}$$

$$= \frac{1}{4\theta^4}E[x^4] - \frac{1}{\theta^3}(2\hat{\theta}) + \frac{1}{\theta^2}$$

$$= \frac{1}{4\theta^8}E(x^4) - \frac{2}{\theta^4} + \frac{1}{\theta^4}$$

$$= \frac{1}{4\theta^8}E(x^4) - \frac{1}{\theta^4}$$

(5160)

$$E[x^4] = \int_0^\infty \frac{x^5}{\theta^2} e^{-\frac{x^2}{2\theta^2}}dx \qquad \hat{x} = \frac{x}{\theta} \qquad dx = \theta\, d\hat{x}$$

$$E[x^4] = \int_0^\infty \frac{(x\theta)^5}{\theta^2} e^{-\frac{x^2}{2}}\theta\, dx$$

$$= \theta^4\int_0^\infty x^5 e^{-\frac{x^2}{2}}dx$$

$$u = \theta^4 x^4 dx \qquad dv = x\, e^{-\frac{x^2}{2}}$$

done in part(c)

$$du = \theta^4\, 3x^3 \qquad V = -e^{-x^2/2}$$

$$E[x^4] = \theta^4 x^4 e^{-x^2/2}\Big|_0^\infty + 3\boxed{\int \theta^4 x^3 e^{-x^2/2}}$$

$$= 0$$

$$dU = 3x^2\theta^4 \qquad dV = x e^{-x^2/2}$$

$$dU = 6x\theta^4 \qquad V = -e^{-x^2/2}$$

$$E[x^4] = 6\theta^4\int_0^\infty x e^{-x^2/2}$$

$$= 6\theta^4 e^{-x^2/2}\Big|_0^\infty = 6\theta^4$$

$$E[(\quad)^2] = \frac{6\theta^4}{4\theta^8} - \frac{1}{\theta^4}$$

$$= \frac{1}{2\theta^4}$$

$$\Rightarrow \text{C.R. BOUND} = \frac{2\theta^4}{n}$$

OVER $\longrightarrow$

3. If $X_1, X_2, \ldots, X_n$ denote a random sample from a $n(\mu, 1)$, find a critical region by using the likelihood ratio principle or the Neyman-Pearson theorem for testing $H_0: \mu = \mu_0$ vs. $H_1: \mu > \mu_0$ $\quad \mu = \mu_1 > \mu_0$

USE NEYMAN PEARSON

$$L(\mu_0) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum^n (x-\mu_0)^2} \qquad L(\mu_1) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum^n (x-\mu_1)^2}$$

$$\lambda = \frac{L(\mu_0)}{L(\mu_1)} = \frac{e^{-\frac{1}{2}\sum^n (x-\mu_0)^2}}{e^{-\frac{1}{2}\sum^n (x-\mu_1)^2}} \gtrless^{H_0}_{H_1} K$$

$$\ln\lambda = -\frac{1}{2}\sum^n (x_i-\mu_0)^2 + \frac{1}{2}\sum^n (x_i-\mu_1)^2 \gtrless \ln K = K'$$

$$= \sum^n (x_i-\mu_1)^2 - \sum^n (x_i-\mu_0)^2 \gtrless 2K' = K''$$

$$= \cancel{\sum^n x_i^2} - 2\mu_1 \sum^n x_i + n\mu_1^2 - \left[\cancel{\sum^n x_i^2} - 2\mu_0 \sum x_i + n\mu_0^2\right] \gtrless 2k''$$

$$2(\mu_0-\mu_1)\sum^n x_i \gtrless^{H_0}_{H_1} 2k'' - n\mu_1^2 + n\mu_0^2 = K'''$$

NOW $\mu_0 < \mu_1 \Rightarrow \mu_0 - \mu_1 < 0$

$$\Rightarrow \sum^n x_i \gtrless^{H_0}_{H_1} \frac{K'''}{2(\mu_0-\mu_1)} = C' \quad \underline{\text{OR}} \quad \bar{X} \gtrless^{H_1}_{H_0} C$$

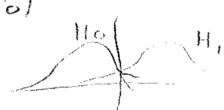
ie, accept H_0 if $\bar{X} < C$
reject H_0 if $\bar{X} \geq C$

(b) Is this a uniformly most powerful critical region?

YES. By restricting the test to a one sided test, this step could be performed which is valid $\forall$ $\mu_1 > \mu_0$ for which this test is Neyman-Pearson optimal

(c) Let $\alpha = .05$ and determine the critical region explicitly.

UNDER H_0 $X_i \sim n(\mu_0, 1) \Rightarrow \bar{X} \sim n(\mu_0, \frac{1}{n})$ $\qquad \alpha = P_r[H_1/H_0]$



$$\frac{\bar{X}-\mu_0}{\sqrt{\frac{1}{n}}}$$

$$\alpha = P_r[\bar{X} \geq C] = P_r\left[(\bar{X}-\mu_0)\sqrt{n} > (C-\mu_0)\sqrt{n}\right]$$

$$= P_r\left[Z > (C-\mu_0)\sqrt{n}\right]$$

$$= 1 - \Phi\left[(C-\mu_0)\sqrt{n}\right]$$

$$(C-\mu_0)\sqrt{n} = 1.645 \Rightarrow C = \frac{\sqrt{n}(1.645)}{\sqrt{n}} + \mu_0$$

4. Let $X \sim n(\mu, 1)$. The best critical region for testing $H_0: \mu = 0$ vs. $H_1: \mu > 0$, based on a sample of size n, is $\{\bar{x};\ \bar{x} > c\}$, where c depends on n.

$\alpha = 0.05$

(a) Find the limiting power under the condition that $\mu = \frac{1}{\sqrt{n}} > 0$.

$\beta = P_r[H_1 / H_1]$

UNDER H_0 $\Rightarrow X \sim n(0,1) \Rightarrow \bar{X} \sim n(0, \frac{1}{n})$

$\alpha = 0.05 = \int_c^{\infty} n(0, \frac{1}{n})dx = 1 - \int_{-\infty}^{c\sqrt{n}} n(0,1)dx$

$= 1 - \Phi(c\sqrt{n})$

$\Rightarrow c\sqrt{n} = 1.645 \Rightarrow c = \boxed{1.645/\sqrt{n}}$

UNDER H_1 $\Rightarrow X_i \sim n(\frac{1}{\sqrt{n}}, \frac{1}{n}) \Rightarrow \bar{X} \sim n(\frac{1}{\sqrt{n}}, \frac{1}{n})$

$\beta = 1 - \int_{-\infty}^{c} n(\frac{1}{\sqrt{n}}, \frac{1}{n}) = 1 - \int^{\frac{c - \frac{1}{\sqrt{n}}}{1/\sqrt{n}}} n(0,1)dx = 1 - \Phi\left[\frac{c - \frac{1}{\sqrt{n}}}{1/\sqrt{n}}\right]$

$1 - \Phi[\boxed{\sqrt{n}\,c} - 1]$

$\lim_{n \to \infty} \beta = 1 - \Phi(\infty) = \cancel{0}$

$-1\frac{1}{2}$

(b) Find the limiting power under the condition that $\mu = \frac{1}{n} > 0$.

$\bar{X} \sim n(\frac{1}{n}, \frac{1}{n}) \Rightarrow \beta = 1 - \int_{-\infty}^{\frac{c - \frac{1}{n}}{1/\sqrt{n}}} n(0,1)dx$

$= 1 - \Phi[\sqrt{n}\,c - \frac{1}{\sqrt{n}}]$

$\lim_{n \to \infty} \beta = 1 - \Phi(\infty) = 0$

$-1\frac{1}{2}$

(c) Determine a sequence $\{\mu_n\}$ such that the limiting power is $\cancel{0}$ 1

$\beta = 1 - \Phi[(c - \mu_n)\sqrt{n}]$

LET $\mu_n = n$

$\beta = 1 - \Phi[c\sqrt{n} - n]$

$-1\frac{1}{2}$

$\lim_{n \to \infty} \beta = 1 - \Phi(-\infty) = 1$

READ FROM p. 145

Let X be a continuous random variable with p.d.f. $f(x)$, and let $X_1, X_2, \ldots, X_n$ be a random sample from $f(x)$, $a < x < b$. Define

$$
\begin{aligned}
&Y_1 = \text{smallest of } (X_1, X_2, \ldots, X_n)\\
&Y_2 = \text{second smallest of } (X_1, X_2, \ldots, X_n)\\
&\quad \vdots\\
(1)\quad &Y_j = j^{th} \text{ smallest of } (X_1, X_2, \ldots, X_n)\\
&\quad \vdots\\
&Y_n = \text{largest of } (X_1, X_2, \ldots, X_n)
\end{aligned}
$$

Thus $Y_1 < Y_2 < \cdots < Y_n$. The Y's are called the n order statistics of the random sample. Y_j is called the j^{th} order statistic of the random sample.

The transformation (1) is not one to one. Since there are $n!$ possible permutations of $(X_1, X_2, \ldots, X_n)$ in increasing order of magnitude, there are $n!$ inverses corresponding to the transformation. Let $A = \{(x_1, x_2, \ldots, x_n);\ a < x_i < b,\ i = 1, 2, \ldots, n\}$ be the sample space corresponding to the random sample and let A_j, $j = 1, 2, \ldots, n!$ be subsets of A corresponding to the $n!$ inverses. Then $A = \bigcup_{j=1}^{n!} A_j$ (A_j – ONE ORDERING) and $A_i \cap A_j = \emptyset$, $\forall i \neq j$. If $B = \{(y_1, y_2, \ldots, y_n);\ a < y_1 < y_2 < \cdots < y_n < b\}$, where the y's are defined by (1), then (1) defines one-to-one transformations which map each of $A_1, A_2, \ldots, A_{n!}$ onto the same set B. The Jacobian for the transformation corresponding to each of the $n!$ inverses is the determinant of the $n \times n$ identity matrix with rows interchanged and is equal to ± 1.

The joint p.d.f. of $Y_1, Y_2, \ldots, Y_n$ is thus

$$g(y_1, y_2, \ldots, y_n) = \sum_{j=1}^{n!} \prod_{i=1}^{n} f(y_i)\,|J_j|$$
$$= n!\, f(y_1) f(y_2) \cdots f(y_n), \quad a < y_1 < y_2 < \cdots < y_n < b.$$

Problems:

(1) Find the p.d.f. of Y_1; of Y_n.

(2) Find the p.d.f. of Y_j.

(3) Find the joint p.d.f. of Y_i and Y_j $(i < j)$.

(4) Find the p.d.f. of the sample median.

(5) Find the p.d.f. of the sample range.

(1) p.d.f. of Y_1

$$G_1(y_1) = P_r\{Y_1 \le y_1\} = P_r\{\min_{1\le i\le n} X_i \le y_1\} = 1 - P_r\{\min_{1\le i\le n} X_i \ge y_1\}$$
$$= 1 - \prod_{i=1}^{n} P_r\{X_i \ge y_1\} = 1 - [1 - F(y_1)]^n$$
$$\Longrightarrow g_1(y_1) = G_1'(y_1) = n[1 - F(y_1)]^{n-1} f(y_1), \quad a < y_1 < b.$$

( ) p.d.f. of Y_n

$$G_n(y_n) = P_r\{Y_n \le y_n\} = P_r\{\max_{1\le i\le n} X_i \le y_n\} = \prod_{i=1}^{n} P_r\{X_i \le y_n\} = [F(y_n)]^n$$
$$\Longrightarrow g_n(y_n) = n[F(y_n)]^{n-1} f(y_n), \quad a < y_n < b.$$

Multinomial distribution (p. 92, text)

$$f(x_1, x_2, \ldots, x_k) = P_r\{X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k)$$
$$= \frac{n!}{x_1!\, x_2! \cdots x_k!}\, p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

where $x_1 + x_2 + \cdots + x_k = n$ and $p_1 + p_2 + \cdots + p_k = 1$.

(2) p.d.f. of Y_j

( ) The p.d.f. of Y_j is

$\mathbb{R}$ $\quad I_1 \quad I_2 \quad I_3$

$(-\infty, Y_i) \quad Y_i \quad Y_i + h \quad (Y_i + h, \infty)$

$(i-1)$ VALUES $\quad$ 1 VALUE $\quad$ $n-i$ SAMPLES

$$g_j(y_j) \overset{\Delta}{=} \lim_{h \to 0} \frac{G_j(y_j + h) - G_j(y_j)}{h}$$

$$= \lim_{h \to 0} \frac{P_r\{ y_j < Y_j \le y_j + h \}}{h}.$$

Let the reals be partitioned as

$$\mathbb{R} = (-\infty, y_j] \cup (y_j, y_j + h] \cup (y_j + h, \infty)$$
$$= I_1 \cup I_2 \cup I_3$$

Let
$$p_1 = P_r\{ X \in I_1 \} = F(y_j)$$
$$p_2 = P_r\{ X \in I_2 \} = F(y_j + h) - F(y_j)$$
$$p_3 = P_r\{ X \in I_3 \} = 1 - F(y_j + h)$$

If $X_1, X_2, \dots, X_n$ denotes a random sample from $f(x)$ then $P_r\{ y_j < Y_j \le y_j + h \}$ is the probability that $j-1$ X fall in I_1, 1 X falls in I_2, and $n-j$ X's fall in I_3.

Thus
$$P_r\{ y_j < Y_j \le y_j + h \} = \frac{n!}{(j-1)!\,1!\,(n-j)!} p_1^{j-1} p_2^{1} p_3^{n-j}.$$

$$= \frac{n!}{(j-1)!(n-j)!} [F(y_j)]^{j-1} [F(y_j + h) - F(y_j)]^1 [1 - F(y_j + h)]^{n-j}.$$

Hence
$$g_j(y_j) = \frac{n!}{(j-1)!(n-j)!} [F(y_j)]^n \lim_{h \to 0} \left\{ \frac{F(y_j + h) - F(y_j)}{h} \right\} [1 - F(y_j + h)]^{n-j}$$

$$= \frac{n!}{(j-1)!(n-j)!} [F(y_j)]^n f(y_j) [1 - F(y_j)]^{n-j}, \quad a < y_j < b.$$

(3) <u>joint p.d.f. of Y_i and Y_j $(i < j)$.</u>

$(\ I_1 \ ](\ I_2 \ ](\ I_3 \ ](\ I_4 \ ](\ I_5 \ )$

$y_i \quad y_i + dy_i \quad y_j \quad y_j + dy_j$

$i-1$ X's $\quad$ 1 X $\quad$ $j-i-1$ X's $\quad$ 1 X $\quad$ $n-j$ X's

The probability $P_r\{y_i < Y_i \le y_i + dy_i,\ y_j < Y_j \le y_j + dy_j\}$ is the probability that $i-1$ X's fall in $(-\infty, y_i]$, 1 X falls in $(y_i, y_i+dy_i]$, $j-i-1$ X's fall in $(y_i+dy_i, y_j]$, 1 X falls in $(y_j, y_j+dy_j]$, and $n-j$ X's fall in (y_j+dy_j, ∞). Thus

$$P_r\{y_i < Y_i \le y_i+dy_i,\ y_j < Y_j < y_j+dy_j\} = g_{ij}(y_i, y_j)\,dy_i\,dy_j$$

$$= \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}\, p_1^{i-1}\, p_2^{1}\, p_3^{j-i-1}\, p_4^{1}\, p_5^{n-j}$$

$$= \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}\left[F(y_i)\right]^{i-1} f(y_i)\,dy_i \left[F(y_j)-F(y_i)\right]^{j-i-1} f(y_j)\,dy_j \left[1-F(y_j)\right]^{n-j}$$

$$\Rightarrow\quad g_{ij}(y_i, y_j) = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}\left[F(y_i)\right]^{i-1}\left[F(y_j)-F(y_i)\right]^{j-i-1}\left[1-F(y_j)\right]^{n-j} f(y_i)\,f(y_j)$$

$$a < y_i < y_j < b.$$

The p.d.f.'s of the sample median and sample range can be found by using $g_i(y_i)$ and $g_{ij}(y_i, y_j)$. This is left to the reader.

REF GIBBON'S (Nonparametric statistics)

Linear Rank Tests

Let $X \sim f_X(x)$, $Y \sim f_Y(y)$; X and Y are continuous random varia.

$X_1, X_2, \ldots, X_m$ is a random sample from $f_X(x)$.

$Y_1, Y_2, \ldots, Y_n$ is a random sample from $f_Y(y)$.

We are interested in statistics for testing the following situations.

(i) $H_0: F_X(x) = F_Y(x)$ for all x
$H_1: F_X(x) \neq F_Y(x)$ for some x

(ii) $H_0: F_X(x) = F_Y(x)$ for all x
$H_1: F_X(x) \geq F_Y(x)$ for all x
$F_X(x) > F_Y(x)$ for some x

(iii) $H_0: F_X(x) = F_Y(x)$ for all x
$H_1: F_Y(x) = F_X(x - \theta)$ for all x, $\theta \neq 0$.

(iv) $H_0: F_X(x) = F_Y(x)$ for all x
$H_1: F_Y(x) = F_X(x\theta)$ for all x, $\theta \neq 1$.

The alternative in (i) is the most general two-sided alternative. The alternative in (ii) is the corresponding one-sided general alternative. The alternative in (iii) is termed a location alternative since it stipulates that X and Y differ only in location. The alternative in (iv) is a scale alternative since it stipulates that X and Y differ only in scale. We will be primarily interested in testing against alternatives in (iii) and (iv). We will be concerned with <u>distribution-free</u> test statistics for testing H_0. These are test statistics which have distributions which do not depend on $F_X(x)$, assuming H_0 is true. A <u>nonparametric test</u> relates to the type of hypothesis being tested. For example, a test in testing (i) or (ii) could be called a nonparametric test since (i) and (ii) are not statements about a parameter. On the other hand (iii) and (iv) involve

... tions and the corresponding tests could not justifiably be termed nonparametric. However, one can still talk about distribution-free tests for testing (iii) and (iv).

Linear Rank Statistic.

Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be random samples from $F_X(x)$ and $F_Y(x)$, respectively. Let $X_{(1)}, X_{(2)}, \ldots, X_{(m)}$ and $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ be the corresponding order statistics. Suppose the two samples are combined and ordered in ascending order. Let us denote the combined ordered sample by a vector of indicator variables as follows. Let

$$Z = (Z_1, Z_2, \ldots, Z_N) \quad , N = n + m,$$

where $Z_i = 1$ if the ith (smallest) random variable in the combined ordered sample is an X and $Z_i = 0$ if it is a Y, for $i = 1, 2, \ldots, N$. For example, if the random samples are $(X_1, X_2, X_3, X_4) = (2, 9, 3, 4)$ and $(Y_1, Y_2, Y_3) = (1, 6, 10)$, the combined ordered sample is $(1, 2, 3, 4, 6, 9, 10)$ or $(Y_1, X_1, X_3, X_4, Y_2, X_2, Y_3)$ and $Z = (0, 1, 1, 1, 0, 1, 0)$. Since $Z_3 = 1$, an X observation (X_3) had rank 3 in the combined ordered sample.

Definition: A linear rank statistic is defined by

$$T_N = \sum_{i=1}^{N} a_i Z_i \quad , \text{where the } a_i \text{ are given numbers.}$$

Note that T_N is linear in the Z_i. No such restriction is implied for the a_i which are called weights or scores.

Properties of Z_i's

Theorem 1: Under $H_0: F_X(x) = F_Y(x)$ for all x, we have, for all $i = 1, 2, \ldots, N$,

$$E(Z_i) = \frac{m}{N}, \quad Var(Z_i) = \frac{mn}{N^2}, \quad cov(Z_i, Z_j) = \frac{-mn}{N^2(N-1)}$$

Proof: $P_r(Z_i = 1) = \frac{m}{N}$, $P(Z_i = 0) = \frac{n}{N}$, $i = 1, 2, \ldots, N$.

Thus $E(Z_i) = \frac{m}{N}$ and $Var(Z_i) = \frac{m}{N} \cdot \frac{n}{N}$.

Also

$$E(Z_i Z_j) = P_r(Z_i = 1, Z_j = 1) \qquad i \neq j$$

$$= \frac{\binom{m}{2}}{\binom{N}{2}} = \frac{m(m-1)}{N(N-1)},$$

and thus

$$cov(Z_i, Z_j) = \frac{m(m-1)}{N(N-1)} - \left(\frac{m}{N}\right)^2 = \frac{-mn}{N^2(N-1)}.$$

Theorem 2: Under $H_0: F_X(x) = F_Y(x)$ for all x

$$E(T_N) = m \sum_{i=1}^{N} \frac{a_i}{N}, \quad Var(T_N) = \frac{mn}{N^2(N-1)} \left[N \sum_{i=1}^{N} a_i^2 - \left(\sum_{i=1}^{N} a_i \right)^2 \right].$$

Proof: $E(T_N) = \sum_{i=1}^{N} a_i E(Z_i) = \sum_{i=1}^{N} a_i \left(\frac{m}{N}\right) = m \sum_{i=1}^{N} \frac{a_i}{N}$.

The proof of the second part is left to the reader. HW

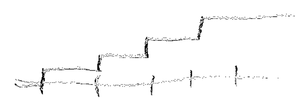
Theorem 3: Let $B_N = \sum_{i=1}^{N} b_i Z_i$ and $T_N = \sum_{i=1}^{N} a_i Z_i$ be two linear rank statistics. Under $H_0: F_X(x) = F_Y(x)$ for all x,

$$cov(B_N, T_N) = \frac{mn}{N^2(N-1)} \left(N \sum_{i=1}^{N} a_i b_i - \left(\sum_{i=1}^{N} a_i \right) \left(\sum_{i=1}^{N} b_i \right) \right).$$

Proof: Left to the reader. HW

Asymptotic Distribution of T_N: We now discuss the limiting distribution of linear rank statistics. The statistic T_N has an asymptotic normal distribution given the appropriate regularity conditions. We need the notion of empirical distribution function. The empirical distribution functions, $S_m(x)$ and $T_n(x)$ corresponding to $X_{(1)}, \ldots, X_{(m)}$ and $Y_{(1)}, \ldots, Y_{(n)}$ are defined as

$$S_m(x) = \begin{cases} 0 & \text{if } x < X_{(1)} \\ \frac{k}{m} & \text{if } X_{(k)} \le x < X_{(k+1)}, \quad k = 1, 2, \ldots, m-1 \\ 1 & \text{if } x \ge X_{(m)} \end{cases}$$

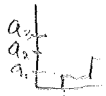


$$T_n(x) = \begin{cases} 0 & \text{if } y < Y_{(1)} \\ \frac{k}{n} & \text{if } Y_{(k)} \le x < Y_{(k+1)} \quad \text{for } k = 1, 2, \ldots, n-1 . \\ 1 & \text{if } x \ge Y_{(n)} . \end{cases}$$

The empirical distribution function for the combined sample is
$H_N(x) = \frac{m}{N} S_m(x) + \frac{n}{N} T_n(x)$. That is $H_N(x)$ is the proportion of observations from either sample which are $\le x$. The statistic $T_N = \sum_{i=1}^{N} a_i Z_i$ can be written as the Stieltjes integral

$$T_N = m \int_{-\infty}^{\infty} J_N [H_N(x)] \, d S_m(x).$$

where $H_N(x) = \lambda_N S_m(x) + (1 - \lambda_N) T_n(x)$, $\lambda_N = \frac{m}{N}$, and $J_N(i/N) = a_i$. For example, let $a_i = i/N$, $J_N(H_N(x)) = H_N(x)$ and



$$T_N = m \int_{-\infty}^{\infty} H_N(x) \, d S_m(x) = \frac{m}{N} \int_{-\infty}^{\infty} [m S_m(x) + n T_n(x)] \, d S_m(x)$$

$= \frac{m}{N} \int_{-\infty}^{\infty} (\#\text{ of obs.} \le x) \cdot (\frac{1}{m}$ if x is the value of an X random var. and 0 otherwise$)$

$$= \frac{1}{N} \sum_{i=1}^{N} i Z_i$$

LINEAR RANK STATISTIC

[illegible] T_N can also be written as

( ) $$T_N = \sum_{\{x;\, p(x)>0\}} J_N[H_N(x)]\, p(x) \qquad dF = f\,dx$$

where $p(x) = \begin{cases} \frac{1}{m} & \text{if } x \text{ is the observed value of an } X \text{ r.v.} \\ 0 & \text{otherwise} \end{cases}$

If the X and Y samples are drawn from (continuous) populations F_X and F_Y, respectively, the combined population distribution function is

$$H(x) = \frac{m}{N} F_X(x) + \frac{n}{N} F_Y(x) = \lambda_N F_X(x) + (1-\lambda_N) F_Y(x); \quad \lambda_N = \frac{m}{N}$$

The asymptotic normality of T_N is given by the following theorem due to Chernoff and Savage (1958, (Annals of Math. Stat., Vol. 29, p. 274).

<u>Theorem 4:</u> Subject to certain regularity conditions, the most important of which is that for $J(H) = \lim_{N\to\infty} J_N(H)$,

$$|J^{(r)}(H)| = \left|\frac{d^r J(H)}{dH^r}\right| \le K|H(1-H)|^{-r-\frac{1}{2}+\delta}, \quad \text{for } r = 0,1,2$$

and some $\delta > 0$ and K any constant which does not depend on $m, n, N, F_X,$ or F_Y, then for $\lambda_N = \frac{m}{N}$ fixed

$$\lim_{N\to\infty} P_r\left(\frac{\frac{T_N}{m} - \mu_N}{\sigma_N} \le t\right) = \Phi(t) = \text{STANDARD NORMAL d.f.}$$

where $$\mu_N = E\left(\frac{T_N}{m}\right) = \int_{-\infty}^{\infty} J(H(x)) f_X(x)\,dx \begin{cases} \text{FOR } F_X = F_Y \\ \text{OR } F_X \ne F_Y \end{cases}$$

(

and

$$N\sigma_N^2 = 2\,\frac{1-\lambda_N}{\lambda_N}\Big\{\lambda_N \iint_{-\infty<x<y<\infty} F_Y(x)[1-F_Y(y)]\,J'(H(x))\,J'(H(y))\, f_X(x) f_X(y)\,dx\,dy$$

$$+(1-\lambda_N)\iint_{-\infty<x<y<\infty} F_X(x)[1-F_X(y)]\,J'(H(x))\,J'(H(y))\, f_Y(x) f_Y(y)\,dx\,dy\Big\}.$$

In the above theorem $\sigma_N^2 = \mathrm{Var}\left(\frac{T_N}{m}\right)$. Also $N \to \infty$ in such a way that $m \to \infty$ and $n \to \infty$ so that $\frac{m}{n}$ remains constant. (Actually we need $0 < \lambda_0 \le \lambda_N \le 1-\lambda_0 < 1$ where $\lambda_0 \le \frac{1}{2}$).

<u>Corollary 4:</u> If X and Y have a common distribution $F(x) = F_X(x) = F_Y(x)$ then

$$\mu_N = \int_0^1 J(u)\,du\,,$$

$$N\lambda_N\sigma_N^2 = (1-\lambda_N)\Big\{\int_0^1 J^2(u)\,du - \Big[\int_0^1 J(u)\,du\Big]^2\Big\}.$$

<u>Proof:</u> We have $H(x) = F(x)$ and thus, from theorem 4

$$\mu_N = \int_{-\infty}^{\infty} J(F(x)) f_X(x)\,dx = \int_{-\infty}^{\infty} J(F(x))\,dF(x) = \int_0^1 J(u)\,du.$$

[DIST. OF df IF UNIFORM ON (0,1)]

Also, PLUGGING IN F_X F_Y IN TOP FORMULA

$$N\lambda_N\sigma_N^2 = 2(1-\lambda_N)\iint_{0<x<y<1} x(1-y)\,J'(x)\,J'(y)\,dx\,dy$$

$$= 2(1-\lambda_N) \iint_{0<u<x<y<v<1} \int_u^v \int_x^v J'(x) J'(y)\, dx\, dy\, du\, dv$$

$$= 2(1-\lambda_N) \iint_{0<u<v<1} \int_u^v \int_x^v J'(x) J'(y)\, dy\, dx\, du\, dv$$

$$= 2(1-\lambda_N) \iint_{0<u<v<1} \int_u^v \left[J(v) - J(x) \right] J'(x)\, dx\, du\, dv$$

$$= 2(1-\lambda_N) \iint_{0<u<v<1} \left[J(v) J(x) - \frac{J^2(x)}{2} \right] \Big|_u^v du\, dv$$

$$= (1-\lambda_N) \iint_{0<u<v<1} \left[J^2(v) - 2J(v)J(u) + J^2(u) \right] du\, dv$$

$$= (1-\lambda_N) \left[\int_0^1 v J^2(v)\, dv + \int_0^1 (1-u) J^2(u)\, du - \int_0^1 J(u)\, du \int_0^1 J(v)\, dv \right]$$

$$= (1-\lambda_N) \left\{ \int_0^1 J^2(u)\, du - \left[\int_0^1 J(u)\, du \right]^2 \right\}$$

The expressions in Corollary 4 are equivalent to those given in Theorem 2 for $a_i = J_N(i/N)$.

<u>Examples:</u> Let $a_i = \frac{i}{N}$. Then $T_N = \sum_{i=1}^{N} \frac{i}{N} Z_i = W_N$, a test statistic due to Wilcoxon. By Theorem 2 we have (under $H_0: F_X(x) = F_Y(x)$)

$$E(W_N) = m \sum_{i=1}^{N} \frac{a_i}{N} = \frac{m}{N} \sum_{i=1}^{N} \frac{i}{N} = \frac{m}{N^2} \frac{(N+1)N}{2} = \frac{m(N+1)}{2N}.$$

$$Var(W_N) = \frac{mn}{N^2(N-1)} \left[N \sum_{i=1}^{N} i^2 - \left(\sum_{i=1}^{N} i \right)^2 \right] \frac{1}{N^2} \qquad \text{(Thm 2)}$$

$$= \frac{\cdots}{\cdots} \left[\frac{n(N+1)(2N+1)}{6} - \frac{n^2(N+1)^2}{N} \right] \frac{1}{N^2}$$

$$= \frac{\cdots (N+1)}{\cdots} \cdot \frac{1}{N^2}$$

Then Cond. ... we have, with $J(u) = u$,

$$\mu_N = E(\cdots) = E\left(\frac{W_N}{m}\right) = \int_0^1 u\, du = \frac{1}{2}$$

$$\Rightarrow E(W_N) = \frac{m}{2} \text{. (which is asymptotically } \frac{m}{2} \cdot \frac{N+1}{N} \text{.)}$$

~~[illegible]~~.

$$N \lambda_N \sigma_N^2 = N \lambda_N \, Var\left(\frac{W_N}{m}\right)$$

$$= (1 - \lambda_N) \left\{ \int_0^1 u^2 du - \left[\int_0^1 u\, du \right]^2 \right\}$$

$$= \frac{n}{N} \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{n}{N} \cdot \frac{1}{12}$$

or $Var(W_N) = \frac{m^2}{m} \cdot \frac{n}{N} \cdot \frac{1}{12} = \frac{mn}{12N}$ which is asymptotically $\frac{mn}{12N}\left(\frac{N+1}{N}\right)$.

<u>Problem:</u> Define $U_{ij} = \begin{cases} 0 & \text{if } Y_j < X_i \\ 1 & \text{if } Y_j > X_i \end{cases}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$,

and $U = \sum_{i=1}^{m} \sum_{j=1}^{n} U_{ij}$. If $W_N = \sum_{i=1}^{N} i Z_i$ show that

$U = W_N - \frac{m(m+1)}{2}$, i.e. U and W_N are linearly related. (U is called the Mann-Whitney statistic.)

Location Alternatives

Consider $H_0: F_X(x) = F_Y(x)$ for all x
against $H_1: F_Y(x) = F_X(x-\theta)$ for all x and $\theta \neq 0$.
θ is a location parameter.
We can consider H_0 to be $H_0: \theta = 0$ and the alternative can be $H_1: \theta \neq 0$, $H_1: \theta > 0$, or $H_1: \theta < 0$.

The distribution function of the Y population is the same as that of the X population but shifted to the left if $\theta < 0$ and shifted to the right if $\theta > 0$. If $\theta < 0$ the median of the X population is larger than the median of the Y population.

We now consider some linear rank statistics that are useful in testing for a shift in location.

$T_N = \sum_{i=1}^{N} a_i Z_i$
IF $\theta > 0$, WE NEED MONO. INCREASING
$X_1 \ldots X_m$, $Y_1 \ldots Y_m$
REJECT $H_0: \theta = 0$
IF T_n IS TO SMALL

<u>Wilcoxon Test:</u> (1945) $\quad W_N = \sum_{i=1}^{N} \frac{i}{N} Z_i$.

We have $\quad E\left(\frac{W_N}{m}\right) = \frac{N+1}{2N} \sim \frac{1}{2}$, $E(W_N) \sim \frac{m}{2}$

and $\quad Var\left(\frac{W_N}{m}\right) = \frac{n(N+1)}{12m} \cdot \frac{1}{N^2}$, $Var(W_N) \sim \frac{mn}{12N}$.

The decision is to reject $H_0: \theta = 0$ if W_N is too large or too small in testing against $H_1: \theta \neq 0$. If W_N is too large one would reject $H_0: \theta = 0$ in favor of [illegible] ~~[illegible]~~ If W_N is too small one would reject $H_0: \theta = 0$ in favor of $H_1: \theta > 0$. $H_1: \theta < 0$.

$a_i = i/N$

$H_1: \theta \neq 0$: REJECT IF W_1 IS TO BIG OR TO SMALL

UNDER $H_0: \theta = 0$, EVERY $(Z_1 \ldots Z_N)$ HAS PROBABILITY $1/\binom{N}{m}$ (under H_0)

Problem: Find the distribution of W_N by enumeration if $m = 3$ and $n = 4$.

SUPPOSE $\exists$ k_1 ORDERINGS $\ni W_1 = W_N$ THEN $P_r[W_N = w_1] = \dfrac{k_1}{\binom{N}{m}}$

SMALLEST $W_N = 6/7$
BIGGEST $W_N =$ [illegible]

Squared Rank Test: $S_N = \sum_{i=1}^{N} \left(\frac{i}{N}\right)^2 Z_i$.

In this case $J_N\left(\frac{i}{N}\right) = \left(\frac{i}{N}\right)^2 \Rightarrow J(u) = u^2$.

Thus, by the Chernoff-Savage theorem, (under H_0),

$$E\left(\frac{S_N}{m}\right) = \int_0^1 u^2 \, du = \frac{1}{3} \Rightarrow E(S_N) \sim \frac{m}{3},$$

and

$$\operatorname{Var}\left(\frac{S_N}{m}\right) = \frac{(n/N)}{m}\left\{\int_0^1 u^4 \, du - \frac{1}{9}\right\}$$

$$= \frac{n}{mN}\left(\frac{1}{5} - \frac{1}{9}\right) = \frac{n}{mN}\left(\frac{4}{45}\right) \Rightarrow \operatorname{Var}(S_n) \sim \frac{nm}{N}\frac{4}{45}.$$

The rejection criteria in this case are the same as for W_N.

Problem: Find the null distribution of ~~W_N~~ S_N by enumeration if $m = 3$ and $n = 4$.

Generally, almost any set of monotone-increasing weights a_i can be used in testing against location

$T = \frac{1}{N}\sum_{i=1}^{N} E(\xi_{(i)}) Z_i$ $\xi_{(i)}$ = i-TH ORDER STAT IN SAMPLE OF SIZE N FOR A $N(0,1)$ (ASYMPTOTICALLY AS GOOD AS T)

alternatives.

Scale Alternatives: $H_0: F_X(x) \equiv F_Y(x)$

$H_1: F_Y(x) \equiv F_X(\theta x), \quad \theta \neq X$

SAME AS FOR TESTING WHEN VARIANCE THEY ONLY

The hypothesis can be stated as $\theta = 1$ and the alternative can be $H_1: \theta \neq 1$, $H_1: \theta > 1$, or $H_1: \theta < 1$.

The distribution function of the Y population is the same as that of the X population but with compressed scale (smaller variance) or enlarged scale (larger variance) according as $\theta > 1$ or $\theta < 1$, respectively. Not just any alternative can be taken exclusively as a scale alternative. For example consider $f_X(x) = e^{-x}$, $x > 0$. Under $H_1: \theta < 1$ we have $f_Y(x) = \theta e^{-\theta x}$ for some $\theta < 1$. Now $Var(X) = 1$ and $Var(Y) = \frac{1}{\theta^2}$ so that $\sigma_Y^2 > \sigma_X^2$. But $E(X) = 1$ and $E(Y) = 1/\theta > E(X)$. Furthermore median$(X) = \ln 2$, median$(Y) = \ln(2/\theta) >$ median(X) for all $\theta < 1$. The combined ordered sample will thus reflect both dispersion and scale differences. The scale alternative should be interpreted as a dispersion alternative only if the population locations are the same or very nearly so. The scale alternative $F_Y(x) = F_X(\theta x)$ is general enough only when the locations are the same for all θ since $E(X) = \theta E(Y) \Longrightarrow$ $E(X) = E(Y) = 0$ or median$(X) =$ median$(Y) = 0$.

In testing for scale we assume that both populations have the same location. A linear rank statistic used in testing for scale must reflect differences in dispersion in the two populations. If the X population has a larger dispersion, the X values should be positioned approximately symmetrically at both extremes of the Y values. Thus the weights a_i should be symmetric, e.g., small weights in the middle and large at the two extremes, or vice versa.

Mood Test: (1954) $$\sum_{i=1}^{N}\left(\frac{i}{N}-\frac{N+1}{2N}\right)^2 Z_i = M_N$$

$$E(M_N)=\frac{m(N^2-1)}{12N^2}, \qquad Var(M_N)=\frac{mn(N+1)(N^2-4)}{180N^4}$$

$$a_i=\frac{i}{N}-\frac{N+1}{2N} \implies J(u)=\left(u-\tfrac{1}{2}\right)^2$$

$$\therefore\ E\left(\frac{M_N}{m}\right)=\int_0^1\left(u-\tfrac{1}{2}\right)^2du=\left.\frac{\left(u-\frac{1}{2}\right)^3}{3}\right|_0^1=\frac{1}{12}$$

$$Var\left(\frac{M_N}{m}\right)=\frac{n/N}{m}\left\{\int_0^1\left(u-\tfrac{1}{2}\right)^4du-\frac{1}{144}\right\}=\frac{n}{mN}\left(\frac{1}{180}\right)$$

Ansari-Bradley Test: (1960)

$$A_N=\sum_{i=1}^{N}\left|\frac{i}{N}-\frac{N+1}{2N}\right|Z_i; \quad a_i=\left|\frac{i}{N}-\frac{N+1}{2N}\right|.$$

Find $E(A_N)$ and $Var(A_N)$ both exactly and by using the Chernoff-Savage theorem.
Also find the null distribution of M_N and A_N when $m=3$ and $n=4$.

If $N = m+n$ is even, order the X's and Y's and assign the ranks as follows:

$Z_{(1)}$	$<$ $Z_{(2)}$	$< \cdots$		$\cdots <$	$Z_{(N-1)}$	$<$ $Z_{(N)}$
1	2	$\cdots$ $\frac{N}{2}$	$\frac{N}{2}$	$\cdots$	2	1

If $N = m+n$ is odd, assign the ranks as follows:

$Z_{(1)}$	$<$ $Z_{(2)}$	$< \cdots$			$\cdots <$ $Z_{(N-1)}$	$<$ $Z_{(N)}$
1	2	$\cdots$ $\frac{N-1}{2}$	$\frac{N+1}{2}$	$\frac{N-1}{2}$ $\cdots$	2	1

If N is odd the central observation $Z_{(\frac{N+1}{2})}$ receives a weight of $\frac{N+1}{2}$.

The corresponding linear rank statistic for these weights is

$$A_N' = \sum_{i=1}^{[\frac{N+1}{2}]} i Z_i + \sum_{i=[\frac{N+1}{2}]+1}^{N} (N+1-i) Z_i$$

$$= \sum_{i=1}^{N} \left(\frac{N+1}{2} - \left| i - \frac{N+1}{2} \right| \right) Z_i$$

Thus, the statistic at the bottom of page 16 can be written according to

$$A_N' = \frac{m(N+1)}{2} - N A_N$$

or $$A_N = \frac{m(N+1)}{2N} - \frac{1}{N} A_N'.$$

The statistic A_N' is the Ansari-Bradley statistic however we use A_N since the two are linearly related and thus equivalent for testing purposes.

Problem: Find the mean and variance of A_N using Theorem 2 and using the Chernoff-Savage Theo.

Comparison of Test Statistics by means of Asymptotic Relative Efficiency

Definition: A test statistic T_N is said to be consistent for testing $H_0: \theta \in \omega$ against $H_1: \theta \in \Omega - \omega$ if for all $\theta \in \Omega - \omega$ the limit of the power function as $N \to \infty$ is 1.

Most test statistics are consistent for testing H_0 against H_1. In what follows it is assumed all test statistics are consistent.

The power of a test, β, depends on α (significance level), $\theta \in \Omega - \omega$, and the sample size N. A means of comparing to test statistics T_N and T_N^* would be to fix α (say $\alpha = .05$) and then compute

$\frac{N^*}{N}$, where N^* and N are the sample sizes needed for T_N^* and T_N to achieve a common preassigned power $\beta_N^*(\theta) = \beta_N(\theta)$, ($\theta$ is fixed in advance). This is not a feasible approach since in many cases it is virtually impossible to compute $\beta(\theta)$, let alone the sample size required for $\beta(\theta) = .9$ (say). Even when β can be computed the process is not feasible since one has to vary α, θ, and $\beta^*(\theta) = \beta(\theta)$.

Consider $H_0: \theta = \theta_0$ vs. $H_1: \theta > \theta_0$ (say). A way out of the above situation is to choose a decreasing sequence of alternatives $\{\theta_N\}$ such that $\lim_{N\to\infty} \theta_N = \theta_0$. We have the following definition of asymptotic relative efficiency (A.R.E.).

<u>Definition</u>: Let $\{T_N\}$ and $\{T_N^*\}$ be two sequences of test statistics, all with the same significance level α. Let $\{N_i\}$ and $\{N_i^*\}$ be two monotonic increasing sequences of positive integers such that

$$\lim_{i\to\infty} \beta_{N_i}(\theta_i) = \lim_{i\to\infty} \beta^*_{N_i^*}(\theta_i) = \gamma$$

where γ is not equal to 0 or 1. Then the A.R.E. of test T relative to T^* is

$$ARE(T, T^*) = \lim_{i\to\infty} \frac{N_i^*}{N_i}$$

if this limit exists and is constant for all sequences $\{N_i\}$ and $\{N_i^*\}$.

Suppose T_N and T_N^* are both consistent size α tests for testing $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$ and suppose T_N and T_N^* have rejection regions of the form $T_N \in R$ for $t_N \geq t_{N,\alpha}$ and $T_N^* \in R$ for $t_N^* \geq t_{N,\alpha}^*$ where $t_{N,\alpha}$ and $t_{N,\alpha}^*$ are chosen so that

$$P_r(T_N \geq t_{N,\alpha} \mid \theta = \theta_0) = \alpha = P_r(T_N^* \geq t_{N,\alpha}^* \mid \theta = \theta_0).$$

Suppose T_N and T_N^* satisfy the following conditions:

(i) $\frac{dE(T_N)}{d\theta}$ exists and is nonzero for $\theta = \theta_0$ and is continuous at $\theta = \theta_0$

(ii) $\exists\ c > 0 \ni \lim_{N \to \infty} \dfrac{dE(T_N)/d\theta \,\big|_{\theta=\theta_0}}{\sqrt{N}\,\sigma_{T_N}\big|_{\theta=\theta_0}} = c.$

(iii) $\exists\ \{\theta_N\}_{N=1}^{\infty} \ni$ for some constant $d > 0$, we have

$$\theta_N = \theta_0 + \frac{d}{\sqrt{N}}, \quad \lim_{N\to\infty} \frac{dE(T_N)/d\theta\,\big|_{\theta=\theta_N}}{dE(T_N)/d\theta\,\big|_{\theta=\theta_0}} = 1,$$

$$\lim_{N\to\infty} \frac{\sigma_{T_N}\big|_{\theta=\theta_N}}{\sigma_{T_N}\big|_{\theta=\theta_0}} = 1.$$

(iv) $\lim_{N\to\infty} P_r\left\{ \dfrac{T_N - E(T_N)\,\big|_{\theta=\theta_N}}{\sigma(T_N)\big|_{\theta=\theta_N}} \leq z \right\} = \Phi(z)$

Theorem 5: Under conditions (i)–(iv) the limiting power of the test T_N is $\lim_{N\to\infty} P_r\{T_N \geq t_{N,\alpha} \mid \theta=\theta_N\} = 1-\Phi(z_\alpha - dc)$

where z_α is that no. such that $1-\Phi(z_\alpha)=\alpha$.

Theorem 6: If T_N and T_N^* are two tests satisfying conditions (i)–(iv), the ARE of T relative to T^* is

$$ARE(T,T^*) = \lim_{N\to\infty}\left\{\frac{\frac{dE(T_N)}{d\theta}\Big|_{\theta=\theta_0} \Big/ \sigma_{T_N}\Big|_{\theta=\theta_0}}{\frac{dE(T_N^*)}{d\theta}\Big|_{\theta=\theta_0} \Big/ \sigma_{T_N^*}\Big|_{\theta=\theta_0}}\right\}^2 .$$

(Recall from the definition on page 19 that ARE was defined by $ARE(T,T^*) = \lim_{i\to\infty} \frac{N_i^*}{N_i}$).

Example: Comparison of the t-test (2-sample) with the Wilcoxon test $W_N = \sum_{i=1}^{N}\left(\frac{i}{N}\right) Z_i$ when testing

$$H_0: F_X(x) = F_Y(x) \text{ vs. } H_1: F_Y(x) = F_X(x-\theta), \begin{Bmatrix} \theta \neq 0 \\ \text{or } \theta > 0 \\ \text{or } \theta < 0 \end{Bmatrix}.$$

$N = n+m$

Assume $\frac{m}{n} = \lambda$ (a constant). Assume the two populations have equal variance σ^2. Then the t-statistic is

$$T_N = \sqrt{\frac{mn}{m+n}}\ \frac{\bar{Y}_n - \bar{X}_m}{S_{m+n}} \quad \text{where} \quad S^2_{m+n} = \frac{mS_1^2 + nS_2^2}{m+n-2}.$$

$$T_N = \sqrt{\frac{mn}{N}} \left(\frac{\bar{Y}_n - \bar{X}_m - \theta}{\sigma} + \frac{\theta}{\sigma} \right) \frac{\sigma}{S_{m+n}} .$$ Now $\frac{S_N}{\sigma} \xrightarrow{P} 1$ as $N \to \infty$ and thus the moments of T_N for N large and $\theta = \mu_Y - \mu_X$ are $E(T_N) = \frac{\theta}{\sigma}\sqrt{\frac{mn}{N}}$ and

$$\sigma^2_{T_N} = \frac{mn}{N} \frac{\sigma^2/n + \sigma^2/m}{\sigma^2} = 1.$$ Therefore

$$\frac{d\,E(T_N)}{d\theta} = \frac{1}{\sigma}\sqrt{\frac{mn}{N}} .$$

Let $T_N^* = W_N = \sum_{i=1}^{N} \left(\frac{i}{N}\right) Z_i$. Then by the Chernoff-Savage Theorem

$$E\left(\frac{T_N^*}{m}\right) = \int_{-\infty}^{\infty} H(x)\, dF_X(x) = \int_{-\infty}^{\infty} \left[\lambda_N F_X(x) + (1-\lambda_N) F_Y(x)\right] dF_X(x)$$

$$= \int_{-\infty}^{\infty} \left[\lambda_N F_X(x) + (1-\lambda_N) F_X(x-\theta)\right] dF_X(x)$$

$$\therefore \left.\frac{d\,E(T_N^*)}{d\theta}\right|_{\theta=\theta_0} = m \int_{-\infty}^{\infty} (1-\lambda_N) f_X(x-\theta)\, dF_X(x) \Big|_{\theta=\theta_0} = \frac{mn}{N}\int_{-\infty}^{\infty} f^2(x)\,dx .$$

Also, from p. 13, $\sigma^2_{T_N^*}\Big|_{\theta=\theta_0} = \frac{mn}{12N}$. Thus

$$ARE(T_N^*, T_N) = \lim_{N\to\infty} \left\{ \frac{\frac{mn}{N}\int_{-\infty}^{\infty} f^2(x)\,dx \Big/ \sqrt{\frac{mn}{12N}}}{\frac{1}{\sigma}\sqrt{\frac{mn}{N}} \Big/ 1} \right\}^2$$

$$= 12\sigma^2 \left\{ \int_{-\infty}^{\infty} f^2(x)\,dx \right\}^2 .$$

We can take $\mu_X = 0$ and $\mu_Y = \theta$. If F_X is the standard normal dist. then $ARE(T^*, T) = 3/\pi$.